OVERCONVERGENT CHERN CLASSES AND HIGHER CYCLE CLASSES

by

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STATEMENT OF DISSERTATION APPROVAL

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ABSTRACT

The goal of this work is to construct integral Chern classes and higher cycle classes for a smooth variety over a perfect field of characteristic $p > 0$ that are compatible with the rigid Chern classes defined by Petrequin. The Chern classes we define have coefficients in the overconvergent de Rham–Witt complex of Davis, Langer and Zink, and the construction is based on the theory of cycle modules discussed by Rost. We prove a comparison theorem in the case of a quasi-projective variety.

RESUME

Le but de ce travail est de construir des classes de Chern entières et des classes de cycles pour une variété lisse sur un corps parfait de caractéristique $p > 0$ compatible aux classes de Chern rigides définies par Petrequin. Les classes de Chern que l'on définit sont à coéfficientes dans le complexe de de Rham–Witt surconvergent de Davis, Langer et Zink et la construction repose sur la théorie de modules de cycles discutée par Rost. On démontre un théorème de comparaison dans le cas d'une variété quasi-projective.

In memoriam Jochen Riedel.

CONTENTS

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CHAPTER 1

INTRODUCTION

It is well known that crystalline cohomology is a good integral model for Berthelot's rigid cohomology in the case of a proper variety. The overconvergent de Rham–Witt cohomology introduced by Davis, Langer and Zink [DLZ11] is an integral p-adic cohomology theory for smooth varieties designed to be compatible with Monsky–Washnitzer cohomology in the affine case and with rigid cohomology in the quasi-projective case.

In view of the fact that for proper smooth varieties over a field of characteristic $p > 0$ crystalline Chern classes are integral Chern classes, which are according to Petrequin compatible with the rigid ones [Pet03], the following question is reasonable:

Question: Can we define integral Chern classes for (open) smooth varieties that are compatible with the rigid ones?

We use the above mentioned overconvergent de Rham–Witt complex as an obvious choice for coefficients for integral Chern classes on smooth varieties.

Let X be a smooth variety over a perfect field k of positive characteristic $p > 0$. Denote by $W^\dagger\Omega_X$ the étale sheaf of overconvergent Witt differentials over X. We construct a theory of higher Chern classes with coefficients in the overconvergent complex

$$c_{ij}^{\mathrm{sc}} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X).$$

If X is quasi-projective we prove the following comparison:

Proposition. *The overconvergent Chern classes* $c_{ij}^{sc} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X)$ *are compatible with the rigid Chern classes* $c_{ij}^{rig} : K_j(X) \to \mathrm{H}_{rig}^{2i-j}(X/K)$ *defined by Petrequin [Pet03] via the comparison morphism of [DLZ11].*

Or more explicitly, the following diagram commutes for all i,j:

$$\begin{tikzcd} & \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X) \otimes \mathbb{Q} \\ K_j(X) \arrow[ur, "c_{ij}^{\mathrm{sc}}"] \arrow[dr, "c_{ij}^{\mathrm{rig}}"'] & \\ & \mathrm{H}^{2i-j}_{\mathrm{rig}}(X/K) \arrow[uu, "\cong"'] \end{tikzcd} \tag{1.1}$$

where the vertical map is the comparison isomorphism.

Let us now present the different parts of the article.

We begin by recalling facts about Milnor K-theory for local rings, including the Gersten conjecture for the associated sheaf on a scheme X where all residue fields have "enough" elements due to Kerz [Ker08]. We note that the Gersten conjecture implies that the "naïve" definition of the Milnor K-sheaf as the sheaf associated to the presheaf given for a ring A by

$$\overline{K}^M_*(A) = T^*(A) / \text{ Steinberg relations}$$

coincides with the definition used by Rost [Ros96] denoted by $\mathscr{K}^M_*$.

In order to be able to apply our results to a more general case, we describe Kerz's "improved" Milnor K-theory. Kerz points out that the usual Milnor K-sheaf $\overline{\mathscr{K}}^M_*$ is continuous and disposes of a natural transfer map if restricted to schemes with infinite residue fields. The improved Milnor K-sheaf $\widehat{\mathscr{K}}^M_*$ is the universal sheaf which is continuous, has a transfer regardless of the residue field and allows a natural transformation

$$\overline{\mathscr{K}}^M_* \mapsto \widehat{\mathscr{K}}^M_*$$

such that for any continuous sheaf G with a transfer together with a natural transformation $\overline{\mathscr{K}}^M_n \to G$ there is a unique natural transformation $\widehat{\mathscr{K}}^M_n \to G$ making the diagram

$$\begin{tikzcd} \overline{\mathscr{K}}^M_n \arrow[rr] \arrow[dr] & & \widehat{\mathscr{K}}^M_n \arrow[dl, dashed, "\exists !"] \\ & G & \end{tikzcd}$$

commute. The sheaf $\widehat{\mathscr{K}}_*^M$ also satisfies the Gersten conjecture and consequently this definition of the Milnor K-sheaf coincides with the definition used by Rost [Ros96].

In Chapter 3 we state Rost's axiomatic approach to Chow groups in terms of cycle modules. We will later use the fact that the Milnor K-ring is a cycle module. An important result is Corollary 3.16, which makes it possible to calculate the cohomology of a cycle module in terms of the associated Chow groups. In particular, we can calculate the Zariski cohomology of the Milnor K-sheaf in terms of the cohomology of the associated cycle complex. We make use of this in the proof of the Projective Bundle Formula (Proposition 3.19), sketched by Gillet in his survey [Gil05]. The statement is proved for Chow groups in general, but is in particular applied to the Milnor K-sheaf in the next chapter in order to show that it provides a duality theory.

In Chapter 4 we start out by giving the axioms of Gillet's generalised duality theories. In Theorem 4.8 and Theorem 4.9 we recall his result that for a duality theory $\Gamma(*)$ satisfying such axioms there exists a theory of higher Chern classes

$$c_{ij} : K_j(X) \to \mathrm{H}^{di-j}(X, \Gamma(i)).$$

We now define a duality theory by setting

$$\underline{\Gamma}_X^*(j) = \mathscr{K}_j^M.$$

As we show that it satisfies Gillet's axioms, we can finally conclude in Theorem 4.10 that there is a theory of Chern classes with coefficients in the Milnor K-sheaf

$$c_{ij} : K_j(X) \to \mathrm{H}^{i-j}(X, \mathscr{K}_n^M).$$

Assume now that k is a perfect field of characteristic $p > 0$ and X a smooth k-scheme. The results of the last two Chapters are for étale topology only.

In Chapter 5 we recall the definition of the overconvergent de Rham–Witt complex $W^\dagger\Omega_X$ introduced by Davis, Langer and Zink in [DLZ11]. It is easy to see that logarithmic Witt differentials are in fact overconvergent. This leads us to define for every $i \geqslant 0$ a morphism

$$\begin{aligned} d\log^i : \mathscr{O}_X^* \otimes \cdots \mathscr{O}_X^* &\to W\Omega^i_{X,\log} \to W^\dagger\Omega_X\,[i] \\ x_1 \otimes \cdots \otimes x_i &\mapsto d\log(x_1)\cdots d\log(x_i). \end{aligned}$$

In Proposition 5.5 we prove that the symbols $d\log(x_1)\cdots d\log(x_i)$ satisfy the Steinberg relation. Therefore the morphism $d\log^i$ factors through the naïve Milnor K-sheaf

$$d\log^i : \overline{\mathscr{K}}_i^M \to W^\dagger\Omega\,[i]\,.$$

We show that the overconvergent de Rham–Witt complex has a transfer map or norm that satisfies the conditions given in [Ker10]. Moreover, it is continuous. As a consequence we obtain for each i a unique natural transformation

$$\widehat{d\log^i} : \widehat{\mathscr{K}}_i^M \to W^\dagger\Omega\,[i]\,,$$

and we do not have to distinguish any more between the different definitions of Milnor K-theory.

This enables us by functoriality of sheaf cohomology to define in Chapter 6 Chern classes with coefficients in the overconvergent complex induced by the ones for Milnor K-theory

Theorem. *There is a theory of Chern classes for vector bundles and higher algebraic K-theory of regular varieties over k with infinite residue fields, with values with coefficients in the overconvergent de Rham–Witt complex:*

$$c_{ij}^{sc} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X).$$

As a preparation for our comparison theorem in Chapter 7 we go over Petrequin's definition of rigid Chern classes and how to calculate them with Čech cocycles. We show that they factor through Milnor K-theory. From now on we assume that X/k is smooth and quasi-projective. In this case Davis, Langer and

Zink construct a rigid-overconvergent comparison morphism. In fact, they show that there is a natural quasi-isomorphism

$$R\Gamma_{\mathrm{rig}}(X) \to R\Gamma(X, W^{\dagger}\Omega_{X/k}) \otimes \mathbb{Q}\,.$$

In the last part of this chapter, we show that the overconvergent Chern classes that we constructed are compatible with Petrequin's rigid Chern classes via this comparison map. This relies on the fact that they both factor through Milnor K-theory, and we have a commutative diagram

$$\begin{tikzcd} & & \mathrm{H}^{2j-i}_{\mathrm{rig}}(X/K) \\ K_j(X) \arrow[r, "c_{ij}^M"] \arrow[urr, "c_{ij}^{\mathrm{rig}}"] \arrow[drr, "c_{ij}^{\mathrm{sc}}"'] & \mathrm{H}^{i-j}(X, \mathscr{K}_i^M) \arrow[ur] \arrow[dr] & \\ & & \mathbb{H}^{2i-j}(X, W^{\dagger}\Omega) \arrow[uu] \end{tikzcd}$$

where the outer triangle leads to the desired diagram (1.1).

In Chapter 8 we construct higher cycle classes using the method of Bloch [Blo86b]. For this we first recall the definition of Bloch's higher Chow groups $\mathrm{CH}^b(X, n)$, which under certain conditions calculates Voevodsky's motivic cohomology. They form together the higher Chow ring $\mathrm{CH}^*(X, \cdot)$ of X, and Bloch establishes further properties useful for a cohomology theory, among other things there is a rational relation with algebraic K-theory, which motivates the construction of higher cycle class maps. Similar to the method used for the Chern classes, we construct first cycle maps for the Milnor K-sheaf

$$\eta_M^{bn} : \mathrm{CH}^b(X, n) \to \mathrm{H}^{b-n}(X, \mathscr{K}_b^M),$$

which satisfy a normalisation property, allow flat pull-backs and are compatible with addition and multiplication, thus giving a homomorphism of rings. This can be done because the target cohomology theory $\mathrm{H}^n(X, \mathscr{K}_b^M)$ satisfies certain properties such as weak purity. We use again the map

$$d\log^i : \mathscr{K}_i^M \to W^{\dagger}\Omega[i]$$

to obtain morphisms of higher cycle classes

$$\eta_{\mathrm{sc}}^{bn} : \mathrm{CH}^b(X, n) \to \mathbb{H}^{2b-n}(X, W^\dagger\Omega^{\geqslant b}).$$

CHAPTER 2

MILNOR K-THEORY

In this section we recall the definition and basic properties of Milnor K-theory for fields and rings. Following [Ker09] we give a definition for the Milnor K-sheaves and state the Gersten conjecture in equicharacteristic.

2.1 Milnor K-theory for fields

We start by recalling the definition of the Milnor K-groups for fields in generators and relations along with some properties.

Let F be a field and $T^*(F)$ the tensor algebra of F. Let I be the two-sided homogeneous ideal in $T^*(F)$ generated by the elements $a \otimes (1-a)$ with $a, 1-a \in F^*$.

Definition 2.1. *The Milnor K-groups of the field F are defined to be*

$$K_n^M(F) := T^n(F)/I.$$

They form a graded ring $K_*^M(F) = T^*(F)/I$. The class of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ is denoted by $\{a_1, \ldots, a_n\}$. Elements of I are usually called Steinberg relations.

The following basic properties are standard.

Lemma 2.2. *The map $K_*^M(-)$ is functorial in the sense that for a field extension $F \hookrightarrow E$ there is a natural homomorphism of graded rings*

$$K_*^M(F) \to K_*^M(E).$$

Lemma 2.3. *The following relations hold:*

- *For $x \in K_n^M(F)$ and $y \in K_m^M(F)$ we have $xy = (-1)^{nm} yx$.*
- *For $a \in F^*$: $\{a, -a\} = \{a, -1\}$.*
- *For $a_1, \ldots, a_n \in F^*$ such that $a_1 + \cdots + a_n$ is either 0 or 1: $\{a_1, \ldots, a_n\} = 0$.*

For a discretely valued field (F, ν) with ring of integers A and prime element π, there exists for each n a unique group homomorphism

$$K_n^M(F) \xrightarrow{\partial} K_{n-1}^M(A/\pi)$$

such that for $u_1, \ldots, u_n \in A^*$

$$\begin{aligned} \partial\{\pi, u_2, \ldots, u_n\} &= \{\overline{u}_2, \ldots, \overline{u}_n\} \\ \partial\{u_1, \ldots, u_n\} &= 0. \end{aligned}$$

By multilinearity of the symbols this is enough to define ∂. The following result is due to Milnor [Mil70].

Proposition 2.4. *For a field F there is a split exact sequence*

$$0 \to K_n^M(F) \to K_n^M(F(t)) \xrightarrow{\partial} \oplus_\pi K_{n-1}^M(F[t]/\pi) \to 0$$

where the sum is over all monic irreducible elements $\pi \in F[t]$.

2.2 The theory for local rings with infinite residue fields

We briefly recall Kerz's discussion of Milnor K-theory in the case when the residue fields have "enough" elements (see [Ker09]).

Definition 2.5. *For a regular semilocal ring R over a field k the Milnor K-groups are given by*

$$K_n^M(R) = \mathrm{Ker}\left(\bigoplus_{x \in R^{(0)}} K_n^M(k(x)) \xrightarrow{\partial} \bigoplus_{y \in R^{(1)}} K_n^M(k(y)) \right).$$

In an attempt to generalise the definition of the Milnor K-ring for fields to arbitrary unital rings, one can define a graded ring in the following way:

Definition 2.6. *For a unital ring R let*

$$\overline{K}_*^M(R) = T^*(R)/J,$$

where J is the two-sided homogeneous ideal generated by the Steinberg relations and elements of the form $a \otimes (-a)$.

If R is a regular semilocal ring over a field, there is a canonical homomorphism of groups

$$\overline{K}_i^M(R) \to K_i^M(R),$$

which is surjective if the base field is infinite (or sufficiently large, as in [Ker09]). Kerz proves that in this case the additional relation $\{a, -a\} = 0$ in the definition is obsolete and that the usual relations hold.

Remark 2.6.1. As pointed out by Kerz, the notion of "sufficiently many elements in the residue fields" of R depends on the context. To be safe one might assume the base field k is algebraically closed or more generally has infinitely many elements. However, many results that are of interest for us hold with the weaker assumption that the number of elements in the residue fields is bounded below by a certain constant.

We want to globalise this to schemes.

Definition 2.7. *Define $\overline{\mathscr{K}}_*^M$ to be the Zariski sheaf associated to the presheaf*

$$U \mapsto \overline{K}_*^M(\Gamma(U, \mathscr{O}_U))$$

on the category of schemes.

Inspired by Definition 2.5 one defines the following.

Definition 2.8. *Let $\mathscr{K}_n^M$ be the sheaf*

$$U \mapsto \operatorname{Ker}\left(\bigoplus_{x \in U^{(0)}} i_{x*} K_n^M(k(x)) \xrightarrow{\partial} \bigoplus_{y \in U^{(1)}} i_{y*} K_n^M(k(y)) \right)$$

on the big Zariski site of regular varieties (schemes of finite type) over a field k, where i_x is the embedding of a point x in U.

One part of the Gersten conjecture for Milnor K-theory is to show that these two definitions coincide. Kato constructed a Gersten complex of Zariski sheaves for Milnor K-theory of a scheme X

$$0 \to \overline{\mathscr{K}}_n^M \to \bigoplus_{x \in X^{(0)}} i_{x*} K_n^M(k(x)) \to \bigoplus_{y \in X^{(1)}} i_{y*} K_n^M(k(y)) \to \cdots \tag{2.1}$$

In [Ros96] Rost gives a proof that this sequence is exact if X is regular and of algebraic type over an arbitrary field k except possibly at the first two places.

Exactness at the second place was shown independently by Gabber and Elbaz-Vincent/Müller-Stach. Finally Kerz proved that the Gersten complex is exact at the first place for X a regular scheme over a field, such that all residue fields contain more than M_n elements. The constant M_n is a natural number that depends on the degree n and can be determined via the construction of a transfer homomorphism (or norm) for Milnor K-theory of rings (see [Ker09, Remark 3.5.8]).

Hence the Gersten conjecture holds for Milnor K-theory in this case.

Theorem 2.9. *The Gersten complex (2.1) for Milnor K-theory is exact if X is a regular scheme over a field where all residue fields contain at least M_n elements.*

In particular, this shows

Corollary 2.10. *Let X be a regular scheme of dimension n over a field with at least M elements, where M is $\max\{M_i \mid 0 \leqslant i \leqslant n\}$. Then*

$$\mathscr{K}^M_* = \overline{\mathscr{K}}^M_*.$$

2.3 The theory for local rings with finite residue fields

As Kerz points out in [Ker10], the Gersten conjecture does not hold in general if we use the same construction of Milnor K-theory for local rings with finite residue fields.

Let $\mathfrak{S}$ be the category of abelian sheaves on the big Zariski site of schemes and $\mathfrak{S}\mathfrak{T}$ the full subcategory of sheaves that admit a transfer map in the sense of Kerz [Ker10]. That is to say for an abelian sheaf F there exist for every finite étale extension of local rings $i : A \to B$ a system of norms

$$\left[N_{B'/A'} : F(B') \to F(A')\right]_{A'}$$

where A' runs over all local A-algebras for which $B' = B \otimes_A A'$ is also local. It satisfies the following properties:

1. **Compatibility**: If $A' \to A''$ is a morphism of local A-algebras such that $B' = B \otimes_A A'$ and $B'' = B \otimes_A A''$ are local as well, the diagram

$$\begin{tikzcd} F(B') \arrow[r] \arrow[d, "N_{B'/A'}"'] & F(B'') \arrow[d, "N_{B''/A''}"] \\ F(A') \arrow[r] & F(A'') \end{tikzcd}$$

commutes.

2. **Restriction to base ring**: If $i' : A' \to B'$ is the induced inclusion, the norm $N_{B'/A'}$ satisfies
$$N_{B'/A'} \circ i'_* = \deg(B/A)\,\mathrm{id}_{F(A')} .$$

Furthermore, let $\mathfrak{S}\mathfrak{T}^\infty$ be the full subcategory of sheaves in $\mathfrak{S}$ which admit norms as described if we restrict the system to local A-algebras A' with infinite (or "big enough," cf. [Ker09]) residue fields.

Example 2.11. *The Milnor K-sheaf $\overline{\mathscr{K}}^M_n$ for every n as defined in [Ker09] is an element of $\mathfrak{S}\mathfrak{T}^\infty$ (cf. [Ker10, Proposition 4]).*

Definition 2.12. *A functor on a category of rings is continuous if it commutes with direct limits. More precisely, it is continuous if for every filtering direct limit of rings*
$$A = \varinjlim A_i$$
the natural homomorphism
$$\varinjlim F(A_i) \to F(A)$$
is an isomorphism.

Example 2.13. *Kerz shows that the Milnor K-sheaf $\overline{\mathscr{K}}^M_*$ is continuous since this is true for the presheaf.*

A main result in Kerz's article [Ker10] is that for a continuous functor $F \in \mathfrak{S}\mathfrak{T}^\infty$ there exists a continuous functor $\widehat{F} \in \mathfrak{S}\mathfrak{T}$ and a natural transformation satisfying a universal property. Namely, for an arbitrary continuous functor $G \in \mathfrak{S}\mathfrak{T}$ together with a natural transformation $F \to G$ there is a unique natural transformation $\widehat{F} \to G$ making the diagram

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & \widehat{F} \arrow[dl, dashed, "\exists!"] \\ & G & \end{tikzcd}$$

commutative. Moreover, for a local ring with infinite residue field, the two functors coincide. It is constructed using rational function rings.

For a commutative ring A let $A(t_1, \dots, t_n)$ be the rational function ring in n variables; that is the ring $A[t_1, \dots, t_n]_S$, where we localise at the multiplicative set S consisting of all polynomials $\sum_I a_I t^I$ such that the ideal generated by the coefficients $a_i \in A$ is the unit ideal. Some useful properties of this ring are

– If A is local with maximal ideal $\mathfrak{m}$, the ring $A(t_1, \dots, t_n)$ is local, too, and $S = A[t_1, \dots, t_n] - \mathfrak{m}_t$, where $\mathfrak{m}_t = \mathfrak{m}\, A[t_1, \dots, t_n]$.
– If $A \subset B$ is a finite étale extension of local rings, there is a canonical isomorphism
$$B \otimes_A A(t_1, \dots, t_n) \xrightarrow{\sim} B(t_1, \dots t_n).$$

Denote by $i : A \to A(t)$ the natural homomorphism and by $i_1, i_2 : A(t) \to A(t_1, t_2)$ the natural homomorphisms sending t to t_1 or t_2, respectively. Now we set $\widehat{F}$ to be the Zariski sheafification of the presheaf
$$A \mapsto \ker \left[F\left(A(t)\right) \xrightarrow{i_{1*} - i_{2*}} F\left(A(t_1, t_2)\right) \right].$$
If F is in $\mathfrak{S}\mathfrak{T}^\infty$, this is indeed in $\mathfrak{S}\mathfrak{T}$ (cf. [Ker10, Proof of Theorem 7]), and it is clear that continuity is preserved.

As a corollary, we obtain an "improved" Milnor K-theory, taking into account that $\overline{\mathscr{K}}^M_n$ is in $\mathfrak{S}\mathfrak{T}^\infty$ and continuous.

Corollary 2.14. *For every $n \in \mathbb{N}$ there exists a universal continuous functor $\widehat{\mathscr{K}}^M_n \in \mathfrak{S}\mathfrak{T}$ and a natural transformation*
$$\overline{\mathscr{K}}^M_n \mapsto \widehat{\mathscr{K}}^M_n$$
such that for any continuous $G \in \mathfrak{S}\mathfrak{T}$ together with a natural transformation $\overline{\mathscr{K}}^M_n \to G$ there is a unique natural transformation $\widehat{\mathscr{K}}^M_n \to G$ such that the diagram
$$\begin{tikzcd}
\overline{\mathscr{K}}^M_n \arrow[rr] \arrow[dr] & & \widehat{\mathscr{K}}^M_n \arrow[dl, dashed, "\exists!"] \\
& G &
\end{tikzcd}$$
commutes.

In the affine case this is denoted by

$$K_*^M \mapsto \widehat{K}_*^M$$

We list some of the important properties, proved in [Ker10, Proposition 10].

1. Let $(A, \mathfrak{m})$ be a local ring. Then $\widehat{K}_1^M(A) = A^\times$.
2. $\widehat{K}_*^M(A)$ has a natural structure as graded commutative ring.
3. For a finite étale extension of local rings $A \to B$ there is a canonical transfer map
$$N_{B/A} : \widehat{K}_n^M(B) \to \widehat{K}_n^M(A).$$
4. The natural map $\widehat{K}_2^M(A) \to K_2(A)$ to Quillen K-theory is an isomorphism.
5. The ring $\widehat{K}_*^M(A)$ is skew symmetric.
6. For $a_1, \ldots, a_n \in A^\times$ with $a_1 + \cdots + a_n = 1$, the image $\{a_1, \ldots, a_n\}$ of $a_1 \otimes \cdots \otimes a_n$ in $\widehat{K}_n^M(A)$ is trivial.
7. For any field F we have $K_*^M(F) = \widehat{K}_*^M(F)$.
8. The natural map $K_n^M(A) \to \widehat{K}_n^M(A)$ is an isomorphism if the residue field of A has "enough" elements (compare [Ker09, Remark 5.8]).
9. There exists a homomorphism from Quillen's K-theory
$$K_n(A) \to \widehat{K}_n^M(A)$$
such that the composition
$$\widehat{K}_n^M(A) \to K_n(A) \to \widehat{K}_n^M(A)$$
is multiplication by $(n-1)!$, and the composition
$$K_n(A) \to \widehat{K}_n^M(A) \to K_n(A)$$
is the Chern class $c_{n,n}$.
10. If (A, I) is a Henselian pair, and $s \in \mathbb{N}$ invertible in A/I, then the map induced by the projection
$$\widehat{K}_n^M(A)/s \to \widehat{K}_n^M(A/I)/s$$
is an isomorphism.

11. Let A be regular, equicharacteristic, F its quotient field and $X = \operatorname{Spec} A$. Then the Gersten conjecture holds, i.e., the Gersten complex

$$0 \to \widehat{K}_n^M(A) \to K_n^M(F) \to \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \cdots$$

12. Let X be a regular scheme containing a field. There is a natural isomorphism

$$\mathrm{H}^n(X, \widehat{K}_n^M) \cong \mathrm{CH}^n(X).$$

13. If A is equicharacteristic of characteristic prime to 2, then the map

$$i_A : \widehat{K}_3^M(A) \to K_3(A)$$

is injective.

14. If A is regular and equicharacteristic, there is a natural isomorphism

$$\widehat{K}_n^M(A) \xrightarrow{\sim} \mathrm{H}_{\mathrm{mot}}(\mathrm{Spec}(A), \mathbb{Z}(n))$$

onto motivic cohomology.

In general, the natural map

$$\overline{\mathscr{K}}_*^M(X) \to \widehat{\mathscr{K}}_*^M(X)$$

is not an isomorphism. For example, we mentioned in property (4) that the improved Milnor K-theory is equal to the Quillen K-theory for any local ring A, $\widehat{K}_2^M(A) = K_2(A)$, which is not true in this generality for the usual Milnor K-theory. An example for this was given by Bruno Kahn in the Appendix to [Kah93]. However, from the fact that $\widehat{\mathscr{K}}_*^M$ satisfies the Gersten conjecture, we can deduce a useful corollary.

Corollary 2.15. *Let X be a smooth scheme with finite residue fields. Then*

$$\mathscr{K}_*^M = \widehat{\mathscr{K}}_*^M,$$

where $\mathscr{K}_*^M$ is as in Definition 2.8.

Another important feature of the improved Milnor K-theory is that it is locally generated by symbols. In other words, its elements satisfy the Steinberg relation. In fact Kerz shows the following theorem.

Theorem 2.16. *Let A be a local ring. Then the map*

$$\overline{K}^M_*(A) \to \widehat{K}^M_*(A)$$

is surjective.

Proof. (IDEA) One can use the transfer map for extensions of local fields of degree 2 and 3 to reduce to the cases $n = 2$ and $n = 1$, whereof both are classical if one takes into account (1) and (4) of the list of properties above. $\square$

2.4 Milnor K-theory on the étale site

Although the improved Milnor K-sheaf was constructed over the big Zariski site of all schemes, we can consider it as a sheaf over the big étale site. In particular, the theory makes sense on the small étale site $X_{\text{ét}}$ of a scheme X.

More precisely, we can define $\overline{\mathscr{K}}^M_*$ over the big étale site as in Definition 2.7 **étale** locally instead of **Zariski** locally. In this case, let $\mathfrak{S}_{\text{ét}}$ be the category of abelian sheaves on the big étale site of all schemes and $\mathfrak{S}\mathfrak{T}_{\text{ét}}$ the full subcategory of sheaves that admit a transfer map in the sense of Kerz [Ker10] as described above. This still makes sense as everything is only defined and described locally. Furthermore, let $\mathfrak{S}\mathfrak{T}^{\infty}_{\text{ét}}$ be the full subcategory of sheaves in $\mathfrak{S}_{\text{ét}}$ which admit a transfer if we restrict the system to local A-algebras A' with infinite (or "big enough," cf. [Ker09]) residue fields. On a similar note, continuity can be defined locally so that the Milnor K-sheaf over the étale site is also continuous.

The theorem now reads

Theorem 2.17. *For a continuous functor $F \in \mathfrak{S}\mathfrak{T}^{\infty}_{\text{ét}}$ there exists a universal continuous functor $\widehat{F} \in \mathfrak{S}\mathfrak{T}_{\text{ét}}$ and a natural transformation $F \to \widehat{F}$. That means, for an arbitrary continuous functor $G \in \mathfrak{S}\mathfrak{T}_{\text{ét}}$ together with a natural transformation $F \to G$ there is a unique natural transformation $\widehat{F} \to G$ making the diagram*

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & \widehat{F} \arrow[dl, dashed, "\exists!"] \\ & G & \end{tikzcd}$$

commutative. Moreover, for a local ring with infinite residue field, the two functors coincide.

Proof. The proof of the Zariski case used in [Ker10, Theorem 7] is purely local. Taking into account that the functors in question are sheafifications of presheaves on the category of local rings and furthermore that the only condition that goes beyond this is the existence of a transfer map for finite étale extensions of local rings, we see that the arguments can be carried over verbatim to the case of the étale site instead of the Zariski site. In fact, this is valid for any (Grothendieck) topology in between the étale and Zariski topology. □

The properties of the improved Milnor K-sheave from Theorem 2.16 and [Ker10, Proposition 10] cited above hold in the case of the étale site equally. In particular, the Gersten complex

$$0 \to \widehat{K}_n^M(A) \to K_n^M(F) \to \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \cdots$$

is exact.

Naturally it follows from Theorem 2.16 that the improved Milnor K-sheaf over the étale site is locally generated by symbols.

CHAPTER 3

CYCLE MODULES

The axiomatisation of cycle modules by Rost provides a powerful framework for modules over the Milnor K-sheaf as it allows to reduce the local case. We recall important definitions and properties and prove a projective bundle formula.

3.1 Definition and properties

Rost defines in [Ros96] first cycle premodules as functors from the category of fields to the category of modules over Milnor K-theory, which have transfer morphisms and residue maps for discrete valuations and satisfy the usual canonical axioms.

Let B be a scheme over a field k. Let $\mathscr{F}(B)$ be the category of fields over B, that means finitely generated fields F over k together with a morphism $\operatorname{Spec} F \to \operatorname{Spec} B$.

Definition 3.1. *A cycle premodule M is a functor*

$$M : \mathscr{F}(B) \to \mathscr{A}\mathscr{B}$$

from the category $\mathscr{F}(B)$ to abelian groups together with a $\mathbb{Z}$-grading (or a $\mathbb{Z}/2\mathbb{Z}$-grading) and the following list of data and rules:

(D1) *For each field extension $\varphi : F \to E$ there is a restriction $\varphi_* : M(F) \to M(E)$ of degree 0.*

(D2) *For each finite extension $\varphi : F \to E$, there is a corestriction $\varphi^* : M(E) \to M(F)$ of degree 0.*

(D3) *For each F the group $M(F)$ is equipped with a left $K_*^M(F)$-module structure respecting the grading.*

(D4) *For a valuation ν on F there is a boundary map $\partial_\nu : M(F) \to M(\kappa(\nu))$ of degree -1.*

For a prime π of the valuation v on F let

$$\begin{array}{rcl} s_v^\pi : M(F) & \to & M(\kappa(v)), \\ \rho & \mapsto & \partial_v(\{-\pi\} \cdot \rho). \end{array}$$

(R1) *For $\varphi : F \to E$, $\psi : E \to L$ one has $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. If φ and ψ are finite, one has in addition $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. If φ is finite, put $R = L \otimes_F E$. For $p \in \operatorname{Spec} R$, let $\varphi_p : L \to R/p$ and $\psi_p : E \to R/p$ be the natural maps and let l_p be the length of the localised ring $R_{(p)}$. Then*

$$\psi_* \circ \varphi^* = \sum_p l_p \cdot (\varphi_p)^* \circ (\psi_p)_*.$$

(R2) *For $\varphi : F \to E$, $x \in K_*^M(F)$, $y \in K_*^M(E)$, $\rho \in M(F)$, $\mu \in M(E)$ one has:*

$$\varphi_*(x\rho) = \varphi_*(x)\varphi_*(\rho),$$

and if φ is in addition finite:

$$\begin{array}{rcl} \varphi^*(\varphi_*(x)y) & = & x\varphi^*(\mu) \\ \varphi^*(y\varphi_*(\rho)) & = & \varphi^*(y)\rho. \end{array}$$

(R3) *Let $\varphi : E \to F$, v a valuation on F, which restricts to a nontrivial valuation w on E with ramification index e. Let $\overline{\varphi} : \kappa(w) \to \kappa(v)$ the induced map. Then*

$$\partial_v \circ \varphi_* = e\overline{\varphi}_* \circ \partial_w.$$

On the other hand if φ is finite and w a valuation on E consider for an extension v of w to F the induced map $\varphi_v : \kappa(w) \to \kappa(v)$. Then

$$\partial_w \circ \varphi^* = \sum_v \varphi_v^* \circ \partial_v.$$

If v is a valuation of F, which is trivial on E, then

$$\partial_v \circ \varphi_* = 0.$$

Let $\overline{\varphi} : E \to \kappa(v)$ be the induced map and π a prime of v, then

$$s_v^\pi \circ \varphi_* = \overline{\varphi}_*.$$

If u is a v-unit and $\rho \in M(F)$, then

$$\partial_v(\{u\}\rho) = -\{\overline{u}\}\partial_v(\rho).$$

Sometimes we use the notation $\varphi_* = r_{E|F}$ *and* $\varphi^* = c_{E|F}$.

Lemma 3.2. *For* $\varphi : F \to E$ *finite*

$$\varphi^*\varphi_* = (\deg\varphi)\,\mathrm{id}\,.$$

If φ *is in addition totally inseparable, one also has*

$$\varphi_*\varphi^* = (\deg\varphi)\,\mathrm{id}\,.$$

Proof. This follows from (**R1**) and (**R2**) (cf. [Ros96, Section 1]). □

One can consider $M(F)$ as a right $K_*^M(F)$-module. The maps ∂ and s are called residue homomorphism and specialisation.

Lemma 3.3. *For a valuation* v *on* F, $x \in K_n^M(F)$, $\rho \in M(F)$, π *a prime of* v, *one has*

$$\begin{aligned} \partial_v(x\rho) &= \partial_v(x)s_v^\pi(\rho) + (-1)^n s_v^\pi\partial_v(\rho) + \{-1\}\partial_v(x)\partial_v(\rho) \\ s_v^\pi(x\rho) &= s_v^\pi(x)s_v^\pi(\rho). \end{aligned}$$

Proof. This follows from (**R3**) (cf. [Ros96, Section 1]). □

Definition 3.4. *A pairing* $M \times M' \to M''$ *of cycle premodules is a system of bilinear maps for all* $F \in \mathscr{F}(B)$

$$M(F) \times M'(F) \to M''(F),$$

which respects the gradings and the above mentioned properties in the obvious way. For more details see [Ros96, Definition 1.2]

A ring structure on a cycle premodule M *is a pairing* $M \times M \to M$, *which induces over each* F *an associative and anticommutative ring structure.*

A homomorphism $\omega : M \to M'$ *of cycle premodules of even respectively odd type is given for each field* F *such that*

(1) $\varphi_*\omega_F = \omega_E\varphi_*$

(2) $\varphi^*\omega_E = \omega_F\varphi^*$

(3) $\{a\}\omega_F(\rho) = \pm\omega_F(\{a\}\rho)$

(4) $\partial_v\omega_F = \pm\omega_{\kappa(v)}\partial_v$

Examples 3.5. *1. Milnor K-theory together with the data*

$$\varphi_* \ , \ \varphi^* \ , \ \textit{multiplication} \ , \partial$$

is a $\mathbb{Z}$-graded cycle premodule over any field k with ring structure [Ros96, Theorem 1.4].

2. *Galois cohomology: Any torsion étale sheaf on B gives rise via Galois cohomology to a cycle premodule over B [Ros96, Remark 1.11].*
3. *Quillen's K-theory over a field is a $\mathbb{Z}$-graded cycle premodule over any field k [Ros96, Remark 1.12]*

To pass from cycle premodules to cycle modules additional data is needed. For a scheme X we write $M(x)$ for $M(\kappa(x))$. The generic point is denoted by ξ. If X is normal, the local ring of X at $x \in X^{(1)}$ is a valuation ring; let $\partial_x : M(\xi) \to M(x)$ be the corresponding residue homomorphism. For $x, y \in X$ we set

$$\partial^x_y = \sum_{z|y} c_{\kappa(z)|\kappa(y)} \circ \partial_z : M(x) \to M(y),$$

if $y \in Z^{(1)}$, where $Z = \overline{\{x\}}$, with z running through the finitely many points lying over y in the normalisation $\widetilde{Z}$. If $y \notin Z^{(1)}$, then $\partial^x_y = 0$.

Definition 3.6. *A cycle module M over k is a cycle premodule which satisfies the following conditions:*

(FD) Finite support of divisors*: Let X be a normal scheme and $\rho \in M(\xi_X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.*

(C) Closedness*: Let X be integral and local of dimension 2. Then*

$$\sum_{x \in X^{(1)}} \partial^x_{x_0} \partial^\xi_x : M(\xi) \to M(x_0),$$

where ξ is the generic and x_0 is the closed point of X.

A morphism of cycle modules is a morphism of cycle premodules.

If X is integral and $(\mathbf{FD})$ holds, we set

$$d = (\partial^\xi_x)_{x \in X^{(1)}} : M(\xi) \to \coprod_{x \in X^{(1)}} M(x).$$

Some properties of cycle modules are mentioned below:

(**H**) **Homotopy property for $\mathbb{A}^1$**: The sequence

$$0 \to M(F) \xrightarrow{r} M(F(u)) \xrightarrow{d} \coprod_{x \in \mathbb{A}^1_{F(u)}} M(x) \to 0$$

is an exact complex where $r = r_{F(u)|F}$.

(**RC**) **Reciprocity for curves**: Let X be a proper curve over F. Then

$$M(\xi) \xrightarrow{d} \coprod_{x \in X_{(0)}} M(x) \xrightarrow{c} M(F)$$

is a complex, i.e., $c \circ d = 0$ with $c = \sum c_{\kappa(x)|F}$.

The properties (**FD**), (**C**), (**H**) and (**RC**) are needed as Rost points out in the sense that

- (**FD**) enables us to write down the differentials d of the cycle complexes $C_*(X; M)$.
- (**C**) guarantees that $d \circ d = 0$.
- (**H**) yields the homotopy property of the Chow groups $A_*(X; M)$.
- (**RC**) is needed for proper push forwards.

Moreover, for a cycle module M over a perfect field k we have the following properties:

(**FDL**) **Finite support of divisors on the line**: Let $\rho \in M(F(u))$. Then $\partial_v(\rho) = 0$ for all but finitely many valuations v of $F(u)$ over F.

(**WR**) **Weak reciprocity**: Let ∂_∞ be the residue map for the valuation of $F(u)|F$ at infinity. Then

$$\partial_\infty(A^0(\mathbb{A}^1_F; M)) = 0.$$

A cycle premodule over a perfect field k is a cycle module if and only if the last two properties hold for all fields F over k. This is true for Milnor and Quillen K-theory. Rost pointed out in [Ros96, Remark 2.4] that Milnor's K-ring over any field k is a basic example for a cycle module, which is the reason why I consider cycle modules in this context.

We will now introduce cycle complexes and basic operations on them.

3.2 Cycle complexes

Let M be a cycle module over X and N one over Y. For subsets $U \subset X$ and $V \subset Y$ and a homomorphism

$$\alpha : \coprod_{x \in U} M(x) \to \coprod_{y \in V} N(y)$$

we write $\alpha_x^y : M(x) \to N(y)$ for the components. If $X = Y$, $U \subset X$ and $\omega : M \to N$ a homomorphism of cycle modules, we define the associated change of coefficients by

$$\begin{aligned} \omega_\# : \coprod_{x \in U} M(x) &\to \coprod_{y \in U} N(y) \\ (\omega_\#)_y^x &= \begin{cases} \omega_{\kappa(x)} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \end{aligned}$$

Definition 3.7. *For a cycle module M over X we define a complex of graded modules with respect to dimension*

$$\begin{aligned} C_p(X; M) &= \coprod_{x \in X_{(p)}} M(x) \\ d &= d_X : C_p(X; M) \to C_{p-1}(X; M) \\ d_y^x &= \partial_y^x \quad \textit{as defined above.} \end{aligned}$$

In a similar way one defines a complex of graded modules with respect to codimension

$$\begin{aligned} C^p(X; M) &= \coprod_{x \in X^{(p)}} M(x) \\ d &= d_X : C^p(X; M) \to C^{p+1}(X; M) \\ d_y^x &= \partial_y^x \quad \textit{as defined above.} \end{aligned}$$

Remark 3.7.1. We can also define a version of the above with support in a closed subscheme $y \to X$. Then define

$$C_p^Y(X; M) = \coprod_{\substack{x \in X_{(p)} \\ x \in Y}} M(x)$$

and

$$C_Y^p(X; M) = \coprod_{\substack{x \in X^{(p)} \\ x \in Y}} M(x).$$

It is not hard to show that $d \circ d = 0$ in both cases so that these are indeed complexes. For morphisms $f : X \to Y$ we are now recalling four important classes of induced maps

$$C_p(X;M) \to C_q(Y;M).$$

1. **Push-forward.** If $f : X \to Y$ is finite, we define

$$\begin{aligned} f_* &: C_p(X;M) \to C_p(Y;M) \\ (f_*)^x_y &= \begin{cases} c_{\kappa(x)|\kappa(y)} & \text{if } y = f(x) \text{ and } [\kappa(x) : \kappa(y)] \leqslant \infty \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2. **Pull-back.** For a morphism $g : Y \to X$ let $s(g) = \max\{\dim(y, Y) - \dim(g(y), X) \mid y \in Y\}$. If $x \in X_{(p)}$, $y \in Y_{(q)}$, $g(y) = x$ and $s(g) \leqslant q - p$, then $y \in Y_x^{(0)}$. Let $\mathscr{A}$ be a coherent sheaf on Y. For $x \in X$ and $y \in Y_x^{(0)}$ we define an integer $[\mathscr{A}, g]^x_y \in \mathbb{Z}$: The localisation $Y_{x,(y)} = \operatorname{Spec}(R)$ of Y_x at y is the spectrum of an artinian ring with residue class field $\kappa(y)$. Let $\widetilde{\mathscr{A}}$ be the pull-back of $\mathscr{A}$ along $Y_{x,(y)} \to Y_x \to Y$, then

$$[\mathscr{A}, g]^x_y := l_R(\widetilde{\mathscr{A}})$$

is the length of $\widetilde{\mathscr{A}}$ over R. This induces a homomorphism in the following manner. For $s \in \mathbb{Z}$ with $s(g) \leqslant s$

$$\begin{aligned} [\mathscr{A}, g, s] &: C_p(X;M) \to C_{p+s}(Y;M) \\ [\mathscr{A}, g, s]^x_y &= \begin{cases} [\mathscr{A}, g]^x_y \cdot r_{\kappa(x)|\kappa(y)} & \text{if } g(y) = x \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $X = \operatorname{Spec} F$ for a field F and

$$0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$$

an exact sequence of coherent sheaves over Y, then by the additivity of length we have

$$[\mathscr{A}', g, s] - [\mathscr{A}, g, s] + [\mathscr{A}'', g, s] = 0.$$

Let $F \to E$ a morphism of fields, X of finite type over F and $g : Y = X \times_F E \to X$ the base change. Then we put

$$g^* = [\mathscr{O}_{Y}, , g, 0]\,. \tag{3.1}$$

A morphism $g : Y \to X$ of finite type over a field is said to have relative dimension s if all fibres are empty or equidimensional of dimension $\dim(g) = s$. (For example: open/closed immersions have $s = 0$.) Define

$$g^* = [\mathscr{O}_Y, g, \dim(g)] .$$

3. **Multiplication with unites.** For global sections $a_1, \ldots, a_n \in \mathscr{O}_X^*$ let

$$\begin{aligned} \{a_1, \ldots, a_n\} : C_p(X; M) &\to C_p(X; M) \\ \{a_1, \ldots, a_n\}_y^x(\rho) &= \begin{cases} \{a_1(x), \ldots, a_n(x)\} \cdot \rho & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This makes $C_p(X; M)$ a module over the tensor algebra of $\mathscr{O}_X^*$. If X is defined over a field F, then $K^* \subset \mathscr{O}_X^*$ and $C_p(X; M)$ becomes a module over K_*F.

4. **Boundary maps.** Let X be of finite type over a field, $i : Y \to X$ a closed immersion and $j : U = X \backslash Y \to X$ the inclusion of the complement. We define the boundary map associated to a so-called boundary triple (Y, i, X, j, U)

$$\partial = \partial_Y^U : C_p(U; M) \to C_{p-1}(Y; M).$$

Although we just defined the four basic maps for the cycle complex with respect to dimension, it is clear that there are similar maps for the codimension-cycle complex (or cocycle complex). Sums of composites of maps of the four basic types are called generalised correspondences. Rost shows in [Ros96, Section 4] compatibilities of the four basic types of maps as desired for a reasonable cycle theory. In particular, we have the following results at our disposal.

Proposition 3.8. *1. Let $f : X \to Y$ and $f' : Y \to Z$ be morphisms of schemes of finite type over a field. Then one has*

$$(f' \circ f)_* = f'_* \circ f_*.$$

2. For two morphisms $g : Y \to X$ and $g' : Z \to Y$ let $s \geqslant s(g)$ and $s' \geqslant s(g')$ and let $\mathscr{A}$ and $\mathscr{A}'$ be coherent sheaves on Y and Z respectively, with $\mathscr{A}'$ flat over Z. Then $s + s' \geqslant s(g) + s(g')$ and

$$\left[g'^* \mathscr{A} \otimes_{\mathscr{O}_Z} \mathscr{A}', g \circ g', s + s'\right] = \left[\mathscr{A}', g', s'\right] \circ [\mathscr{A}, g, s] .$$

In particular, for morphisms g and g' as in the formula (3.1) where g' is in addition flat, we have

$$(g \circ g')^* = g'^* \circ g^*.$$

3. *For the diagram*

$$\begin{tikzcd} U \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

with f and f' morphisms of schemes of finite type over a field, let $s \geqslant s(g), s(g')$ and $\mathscr{A}$ a coherent sheaf on Y. Then

$$[\mathscr{A}, g, s] \circ f_* = f'_* \circ \left[f'^* \mathscr{A}, g', s\right].$$

In particular for g as in (3.1)

$$g^* \circ f_* = f'_* \circ g'^*.$$

Proof. This is [Ros96, Proposition 4.1]. □

Lemma 3.9. *Let $f : X \to Y$ a morphism of schemes of finite type over a field.*

1. *If a is a unit on X, then*

$$f_* \circ \{f^*(a)\} = \{a\} \circ f_*.$$

2. *Let f be finite and flat and let a be a unit on Y. Then*

$$f_* \circ \{a\} \circ f^* = \left\{\tilde{f}_*(a)\right\},$$

where $\tilde{f}_* : \mathscr{O}_Y^* \to \mathscr{O}_X^*$ is the standard transfer map. This is often referred to as the projection formula.

Proof. This is [Ros96, Lemma 4.2]. □

Lemma 3.10. *Let a be a unit on X.*

1. *For a morphism of schemes of finite type over a field $g : Y \to X$ of constant codimension as explained above, one has*

$$g^* \circ \{a\} = \{g^* a\} \circ g^*.$$

2. *For a boundary triple (Y, i, X, j, U) one has*

$$\partial_Y^U \circ \{j^*(a)\} = -\{i^*(a)\} \circ \partial_Y^U.$$

Proof. This is [Ros96, Lemma 4.3] □

For a morphism $h : x \to X'$ of schemes of finite type over a field, a closed immersion $Y' \hookrightarrow X'$ and the complement $U' = X' \backslash Y'$ consider the pullback diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\overline{h}"'] & X \arrow[d, "h"] & U \arrow[l, hook'] \arrow[d, "\overline{\overline{h}}"] \\ Y' \arrow[r, hook] & X' & U' \arrow[l, hook'] \end{tikzcd}$$

Proposition 3.11. 1. *If h is proper*

$$\overline{h}_* \circ \partial_Y^U = \partial_{Y'}^{U'} \circ \overline{\overline{h}}_*.$$

2. *If h is flat (of constant relative dimension), then*

$$\overline{h}^* \circ \partial_{Y'}^{U'} = \partial_Y^U \circ \overline{\overline{h}}^*.$$

Proof. This is [Ros96, Proposition 4.4] □

Finally we have

Lemma 3.12. *Let $g : Y \to X$ be a smooth morphism of schemes of finite type over a field of constant dimension 1, let $\sigma : X \to Y$ be a section and let $t \in \mathscr{O}_Y$ be a global parameter defining the subscheme $\sigma(X) \subset Y$. Denote by $\tilde{g}$ the restriction of g to $Y \backslash \sigma(X)$, and ∂ the boundary map associated to σ. Then*

$$\partial \circ \tilde{g}^* = 0 \qquad \text{and} \qquad \partial \circ \{t\} \circ \tilde{g}^* = (\mathrm{id}_X)_*.$$

Proof. This is [Ros96, Lemma 4.5] □

The grading on M induces a grading on the dimension and codimension complex via

$$\begin{aligned} C_p(X;M,q) &= \coprod_{x\in X_{(p)}} M_{q+p}(x) \\ C^p(X;M,q) &= \coprod_{x\in X^{(p)}} M_{q-p}(x). \end{aligned}$$

The maps to be considered will respect the grading.

3.3 The cohomology of cycle modules

The following results are useful if one is faced with the task to calculate the cohomology of a cycle module explicitly.

Definition 3.13. *The Chow group of p-dimensional cycles with coefficients in M is defined as the p^{th} homology group of the complex $C_*(X;M)$*

$$A_p(X;M) := \mathrm{H}_p(C_*(X;M)).$$

Similarly we define the Chow groups

$$\begin{aligned} A^p(X;M) &:= \mathrm{H}^p(C^*(X;M)) \\ A_p(X;M,n) &:= \mathrm{H}_p(C_*(X;M,n)) \\ A^p(X;M,n) &:= \mathrm{H}^p(C^*(X;M,n)). \end{aligned}$$

The morphisms induced by the four basic maps on the cycle complexes induce maps on the homology and cohomology groups and commute, respectively anticommute with the differentials. The compatibilities, for example the ones mentioned in Propositions 3.8 and 3.11 and Lemmata 3.9, 3.10 and 3.12 carry over from cycle modules to Chow groups (for proper f, f' and flat g). Moreover, a boundary triple (Y,i,X,j,U) induces a **long exact sequence for homology**

$$\cdots \xrightarrow{\partial} A_p(Y;M) \xrightarrow{i_*} A_p(X;M) \xrightarrow{j_*} A_p(U;M) \xrightarrow{\partial} A_{p-1}(Y;M) \xrightarrow{i_*} \cdots. \tag{3.2}$$

Recall that the classical Chow groups of p-dimensional cycles may be defined as the cokernel of the divisor map

$$\mathrm{CH}_p(X) := \mathrm{Coker}\left(\coprod_{x\in X_{(p+1)}} \kappa(x)^* \to \coprod_{x\in X_{(p)}} \mathbb{Z} \right).$$

Similarly for the codimension p cycles

$$\mathrm{CH}^p(X) := \mathrm{Coker}\left(\coprod_{x\in X^{(p-1)}} \kappa(x)^* \to \coprod_{x\in X^{(p)}} \mathbb{Z}\right),$$

whereas the ungraded Chow group is

$$\mathrm{CH}(X) := \mathrm{Coker}\left(\coprod_{x\in X} \kappa(x)^* \to \coprod_{x\in X} \mathbb{Z}\right),$$

where the target is isomorphic to the group of all cycles on X. Thus one has the following equalities

$$\begin{aligned} A_p(X; K_*, -p) &= \mathrm{CH}_p(X) \\ A^p(X; K_*, p) &= \mathrm{CH}^p(X) \end{aligned}$$

Another interesting feature of the Chow groups as defined here, which brings it closer to classical topology, is the **homotopy invariance**. Let $\pi : V \to X$ be an affine bundle of dimension n. Then

$$\pi^* : A_p(X; M) \to A_{p+n}(V; M) \tag{3.3}$$

is bijective for all p. If X is equidimensional, then

$$\pi^* : A^p(X; M) \to A^p(V; M) \tag{3.4}$$

is bijective for all p. In particular this applies to fiberproducts with $\mathbb{A}^n$. Rost proves this in [Ros96, Proposition 8.6] using a spectral sequence argument.

We now come to one of the main results that we need from Rost's discussion for our purpose.

Let M be a cycle module over a field k.

Theorem 3.14. *Let X be smooth, semilocal and a localisation of a separated scheme of finite type over k. Then*

$$A^p(X; M) = 0 \qquad \textit{for} \qquad p > 0.$$

Proof. This is Theorem 6.1 in Rost's paper. He notes that this result is known for Quillen's K-theory, étale cohomology as well as for Milnor's K-theory (which is really all we need for our purposes). The main step in the proofs is called Quillen's trick and can be modified with a method due to Panin to fit the general case. $\square$

In other words, the complex $C^*(X;M)$ is acyclic. It is clear that one deduces a similar statement from this for the graded complexes.

When X is smooth, it is possible to sheafify the notion of cycle modules as follows.

Definition 3.15. *Let $\mathscr{M}_X$ be the Zariski sheaf on X given by*

$$U \mapsto A^0(U;M) \subset M(\xi_X),$$

and similarly let $\mathscr{M}_q$ be the Zariski sheaf associated to

$$U \mapsto A^0(U;M,q) \subset M_q(\xi_X).$$

In the case, when $M = K_*^M$ is Milnor K-theory, this definition coincides with the Milnor K-sheaf as defined in 2.8.

Corollary 3.16. *For a smooth variety X over k there are natural isomorphisms*

$$A^p(X;M) = \mathrm{H}^p(\mathrm{X}, \mathscr{M}_X).$$

Proof. Let $\mathscr{C}^p$ be the Zariski sheaf on X associated to $C^p(\cdot;M)$. The complex

$$0 \to \mathscr{M}_X \to \mathscr{C}^0 \to \mathscr{C}^1 \to \cdots$$

is a resolution of $\mathscr{M}_X$. Indeed, the complex is exact in the first and second spot by definition of $\mathscr{M}_X$ and $\mathscr{C}^0$. Exactness at the remaining spots follow from Theorem 3.14. The sheaves $\mathscr{C}^p$ are clearly flasque; thus the assertion follows. $\square$

A special case of this is that under the same conditions as in the corollary we have

$$A^p(X;M,q) = \mathrm{H}^p(X, \mathscr{M}_q).$$

This tells us that we can calculate the Zariski cohomology of the Milnor K-sheaf $\mathscr{K}_*^M$ in terms of the cohomology of the associated cycle complex, which will prove to be a useful tool.

Rost mentions that the resolution of $\mathscr{M}_X$ used in the proof of the corollary has nice functorial properties, which are for example used to construct pull-back maps for complexes or to define pairings of complexes. Among these last, we want to mention a particular case. Let M and N be cycle modules over k; more precisely we consider the cases $M = N$ or $N = K_*$. In later applications the case $M = N = K_*$ will be of particular interest. This is a significant simplification as the notion of a tensor product for cycle modules does not seem to be obvious.

Definition 3.17. *Let Y and Z be schemes of finite type over k. We define the cross product*

$$\times : C_p(Y; N) \times C_q(Z; M) \to C_{p+q}(Y \times Z; M)$$

in the following way: For $y \in Y$ let Z_y be the fibre over y, let $\pi_y : Z_y \to Z$ be the projection and $i_y : Z_y \to Y \times Z$ the inclusion. For $z \in Z$ we make similar definitions with respect to Y. The following definitions are equivalent:

$$\begin{aligned} \rho \times \mu &= \sum_{y \in Y_{(p)}} (i_y)_* \left(\rho_y \cdot \pi_y^*(\mu) \right), \\ \rho \times \mu &= \sum_{z \in Z_{(q)}} (i_z)_* \left(\pi_z^*(\rho) \cdot \mu_z \right). \end{aligned}$$

This makes sense because by assumption we are given a pairing $N \times M \to M$. In particular, the product is understood after pointwise restriction. The map

$$(i_y)_* : C_q(Z_y; M) \to C_{p+q}(Y \times Z; M)$$

is induced by $Z_{y(q)} \subset (Y \times Z)_{(p+q)}$ and it is similar for i_z. It is easy to see that the two definitions coincide as it is symmetric in Z and Y. The following four properties endow the definition with a “good” product structure for complexes.

1. ASSOCIATIVITY. If we have in addition to the data above a scheme X of finite type over k, and $\eta \in C_r(X; N)$, then

$$\eta \times (\rho \times \mu) = (\eta \times \rho) \times \mu.$$

2. COMMUTATIVITY. If $M = N$ is a cycle module with ring structure over k, let $\tau : Y \times Z \to Z \times Y$ be the interchange of factors. Then one has for $\rho \in C_p(Y; M, n)$ and $\mu \in C_q(Z; M, m)$

$$\tau_*(\rho \times \mu) = (-1)^{nm} \mu \times \rho \in C_{p+q}(Z \times Y; M, n + m).$$

3. Chain rule. For $\rho \in C_p(Y; M, n)$ and $\mu \in C_q(Z; M, m)$ one has
$$d(\rho \times \mu) = d(\rho) \times \mu + (-1)^n \rho \times d(\mu).$$

4. Compatibility. The cross product is compatible with the four basic types of maps: push-forward, pull-back, multiplication with units and boundary maps as described earlier.

A special case that carries over to Chow groups is when Y and Z coincide. Thus let X be smooth over k and τ a choice of coordinate for the tangent space TX. One sets

$$\begin{aligned} I_X : C^*(X; N) \times C^*(X; M) &\to C^*(X; M) \\ I_X(\rho, \mu) &= (\rho(\tau) \circ J(X \times X, X))(\rho \times \mu), \end{aligned}$$

where J is induced by the closed immersion $X \to X \times X$ and can be thought of as pull-back along a tubular neighbourhood. By 3 this is a pairing of complexes, and it induces a pairing of Chow groups

$$\cup : A^*(X; N) \times A^*(X; M) \to A^*(X; M).$$

Rost concludes with the following result [Ros96, Theorem 14.6].

Theorem 3.18. *If $M = N$ is a cycle module with ring structure over k, the pairing $\cup$ turns $A^*(X; M)$ into an anticommutative associative ring. If $N = K_*^M$, the pairing turns $A^*(X; M)$ into a module over $A^*(X; K_*^M)$.*

A special case of this is when $M = K_r^M$ and $N = K_s^M$. In that case the above discussion makes clear that for smooth schemes X and Y of dimension n over k the natural product

$$A^*(X; K_*^M) \otimes A^*(Y; K_*^M) \to A^*(X \times Y; K_*^M)$$

is an external product. If X and Y are quasi-projective, we embed them into smooth schemes over k, and the natural product in these smooth schemes descends to a cross product in X and Y, as it is defined pointwise.

3.4 Projective bundle formula for Chow groups

Following Gillet's axiomatic framework to construct Chern classes, one of the main steps is to establish a projective bundle formula. We give here a more detailed proof of the sketch in [Gil05, Proposition 54]

Proposition 3.19. *Let M be a cycle module, X a variety over k and $\pi : \mathscr{E} \to X$ a vector bundle of constant rank n. Let further $\xi \in \mathrm{H}^1(\mathbb{P}(\mathscr{E}), \mathscr{O}^*)$ be the class of $\mathscr{O}(1)$. Then there is an isomorphism*

$$A^p(\mathbb{P}(\mathscr{E}), M, q) \cong \bigoplus_{i=0}^{n-1} A^{p-i}(X, M, q-i)\xi^i.$$

Proof. This being a local question, we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine and that $\mathscr{E} = \mathscr{O}_X^n$. The first part of the proof establishes the result for the case of a point $X = \operatorname{Spec} k$. From there, the second part deduces the general result.

Now let $X = \operatorname{Spec} k$ be a point. This implies in particular that $\mathbb{P}(\mathscr{E}) = \mathbb{P}^n_k$. The formula we have to show in this case reads

$$A^p(\mathbb{P}^n, M, q) = A^0(X, M, q-p)\xi^p$$

because obviously the higher Chow groups vanish for a point so that we are left with only one term with $i = p$. Let $j : \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be the hyperplane at infinity, $\mathbb{A}^n$ its complement and $i : \mathbb{A}^n \to \mathbb{P}^n$ the inclusion of the open subset. Recalling the definition of cycle complexes as

$$\begin{aligned}
C^{p-1}(\mathbb{P}^{n-1}; M, q-1) &= \prod_{x \in (\mathbb{P}^n)^{(p-1)}} M_{q-1-(p-1)}(x) = \prod_{x \in (\mathbb{P}^n)^{(p-1)}} M_{q-p}(x) \\
C^p(\mathbb{P}^n; M, q) &= \prod_{x \in (\mathbb{P}^n)^{(p)}} M_{q-p}(x) \\
C^p(\mathbb{A}^n; M, q) &= \prod_{x \in (\mathbb{A}^n)^{(p)}} M_{q-p}(x)
\end{aligned}$$

together with the fact that points of codimension $p-1$ in $\mathbb{P}^{n-1}$ correspond to points of codimension p in $\mathbb{P}^n$, we see that the maps i and j induce via push-forward and pull-back respectively morphisms

$$j_* : C^{p-1}(\mathbb{P}^{n-1}; M, q-1) \to C^p(\mathbb{P}^n; M, q)$$

and

$$i^* : C^p(\mathbb{P}^n; M, q) \to C^p(\mathbb{A}^n; M, q).$$

By choice and definition of $\mathbb{P}^{n-1}$ and $\mathbb{A}^n$, these morphisms of groups for varying p combine to a short exact sequence of complexes

$$0 \to C^*(\mathbb{P}^{n-1}; M, q-1)\,[1] \to C^*(\mathbb{P}^n; M, q) \to C^*(\mathbb{A}^n; M, q) \to 0,$$

which gives rise to a long exact sequence of Chow groups by taking cohomology

$$\cdots \to A^{p-1}(\mathbb{P}^{n-1}; M, q-1) \xrightarrow{j_*} A^p(\mathbb{P}^n; M, q) \xrightarrow{i^*} A^p(\mathbb{A}^n; M, q) \to \cdots,$$

where j_* is the Gysin map. Let $\pi : \mathbb{P}^n \to X = \operatorname{Spec} k$ be the projection induced from $\pi : \mathscr{E} \to X$. Consequently the map associated to $\pi \cdot i$ on Chow groups

$$(\pi \cdot i)^* : A^p(\operatorname{Spec}(k), M, q) \to A^p(\mathbb{A}^n, M, q)$$

is an isomorphism due to homotopy invariance (3.4). Since the Chow groups of a point are trivial for $p > 0$, the same holds true for the Chow groups of $\mathbb{A}^n$. Therefore we can break up the long exact sequence. The first part reads

$$0 \to A^0(\mathbb{P}^n; M, q) \xrightarrow{i^*} A^0(\operatorname{Spec} k, M, q) \to A^0(\mathbb{P}^{n-1}; M, q-1) \xrightarrow{j_*} A^1(\mathbb{P}^n, M, q) \to 0.$$

Per definitionem

$$A^0(\operatorname{Spec} k, M, q) = C^0(\operatorname{Spec} k; M, q) = M_q(k)$$

and

$$\begin{aligned} A^0(\mathbb{P}^n, M, q) &= \operatorname{Ker}\Big(C^0(\mathbb{P}^n; M, q) \to C^1(\mathbb{P}^n; M, q)\Big) \\ &= \operatorname{Ker}\left(\prod_{x \in (\mathbb{P}^n)^0} M_q(x) \to \prod_{x \in (\mathbb{P}^n)^1} M_{q-1}(x)\right), \end{aligned}$$

and the fact that $(\pi \cdot i)^* = i^* \cdot \pi^*$ is an isomorphism shows that π^* splits the sequence as i^* is injective. Thus

$$A^0(\mathbb{P}^n, M, q) \cong A^0(\mathbb{A}^n, M, q).$$

The other parts of the long exact sequence become for every $i \geqslant 1$

$$0 \to A^{i-1}(\mathbb{P}^{n-1}, M, q-1) \xrightarrow{j_*} A^i(\mathbb{P}^n, M, q) \to 0;$$

hence we have isomorphisms where the map j_* is the same as cap product with ξ. By induction it follows that the natural map

$$\xi^p : A^0(\operatorname{Spec} k, M, q-p) \to A^p(\mathbb{P}^n, M, q)$$

is an isomorphism.

Assume now that $X = \operatorname{Spec} A$ for a k-algebra A. The projective bundle $\mathbb{P}(\mathscr{E})$ takes the form $\mathbb{P}^n_X = \mathbb{P}^n_k \times X$, and it is useful to keep the following commutative diagram of the fibre product in mind

$$\begin{tikzcd}
\mathbb{P}^n_X \arrow[r, "f'"] \arrow[d, "\pi'"'] & \mathbb{P}^n_k \arrow[d, "\pi"] \\
X \arrow[r, "f"'] & \operatorname{Spec} k
\end{tikzcd} \tag{3.5}$$

Let ξ be again the image of the twisting sheaf $\mathscr{O}_{\mathbb{P}^n}(1)$. Cup product with ξ^i for $0 \leqslant i \leqslant n-1$ provides a natural morphism of cohomology

$$\oplus_{i=0}^{n-1}\xi^i : \bigoplus_{i=0}^{n-1} A^{p-i}(X, M, q-i) \to A^p(\mathbb{P}(\mathscr{E}), M, q),$$

and the task is to show that this is an isomorphism. To this end we use the fact that $A^p(X; M, q) = \mathrm{H}^p(X, \mathscr{M}_q)$. Note that $\mathrm{H}^p(\mathbb{P}_X, \mathscr{M}_q) = \mathrm{R}^p\,\Gamma_{\mathbb{P}^n_X}\,\mathscr{M}_q$ and $\mathrm{H}^{p-i}(X, \mathscr{M}_{q-i}) = \mathrm{R}^{p-i}\,\Gamma_X \circ \mathrm{R}^i\,\pi'_*(\mathscr{M}_q)$. Thus the morphism above given by successive multiplication with ξ^i's induces a morphism

$$\mathbf{R}\,\Gamma_X\,\mathbf{R}\,\pi'_*\,\mathscr{M}_q \to \mathbf{R}\,\Gamma_{\mathbb{P}^n_X}\,\mathscr{M}_q$$

in the derived category of abelian groups. Since $\Gamma_X = \Gamma_k \circ f_*$ where we write for simplicity $\Gamma_k = \Gamma_{\operatorname{Spec} k}$, there is a spectral sequence

$$\mathrm{R}^i\,\Gamma_k \circ \mathrm{R}^j\,f_* \Rightarrow R^n\Gamma_X,$$

which degenerates because $\mathrm{R}^i\,\Gamma_k = 0$ if $i \neq 0$. Therefore there is an isomorphism

$$\oplus_{i+j=p}\,\mathrm{R}^i\,\Gamma_X \circ \mathrm{R}^j\,\pi'_*(\mathscr{M}_q) \cong \oplus_{i+j=p}\Gamma_k(\mathrm{R}^i\,f_* \circ R^j\pi'_*)(\mathscr{M}_q),$$

which is in the derived category

$$\mathbf{R}\,\Gamma_X\,\mathbf{R}\,\pi'_*(\mathscr{M}_q) \cong \Gamma_k\,\mathbf{R}\,f_*\,\mathbf{R}\,\pi'_*(\mathscr{M}_q).$$

For the derived functors of the compositions $f_* \circ \pi'_*$ and $\pi_* \circ f'_*$, there are as usual two spectral sequences

$$\mathrm{R}^i\,f_*\,\mathrm{R}^j\,\pi'_* \Rightarrow \mathrm{R}^n(f_* \circ \pi'_*) \quad\text{and}\quad \mathrm{R}^i\,\pi_*\,\mathrm{R}^j\,f'_* \Rightarrow \mathrm{R}^n(\pi_* \circ f'_*),$$

yet the commutativity of the diagram (3.5) implies that they converge in fact to the same object. This in turn leads to an isomorphism in the derived category

$$\Gamma_k\,\mathbf{R}\,f_*\,\mathbf{R}\,\pi'_*(\mathscr{M}_q) \cong \Gamma_k\,\mathbf{R}\,\pi_*\,\mathbf{R}\,f'_*(\mathscr{M}_q).$$

If we recall that push-forward is well defined for cycle modules and compatible with the structure and therefore transforms cycle modules into cycle modules, we see that by the result of the first part of the proof the right-hand-side of this is isomorphic to

$$\mathbf{R}\,\Gamma_{\mathbb{P}^n_k}\,\mathbf{R}\,f'_*(\mathscr{M}_q).$$

Now similarly to above, the spectral sequence associated to the equality of functors $\Gamma_{\mathbb{P}^n_X} = \Gamma_{\mathbb{P}^n_k} \circ f'_*$ induces an isomorphism in the derived category

$$\mathbf{R}\,\Gamma_{\mathbb{P}^n_k}\,\mathbf{R}\,f'_*(\mathscr{M}_q) \cong \mathbf{R}\,\Gamma_{\mathbb{P}^n_X}\,\mathscr{M}_q\,.$$

Putting everything together yields an isomorphism

$$\mathbf{R}\,\Gamma_X\,\mathbf{R}\,\pi'_*(\mathscr{M}_q) \cong \mathbf{R}\,\Gamma_{\mathbb{P}^n_X}\,\mathscr{M}_q,$$

which corresponds by construction exactly the morphism of cohomology introduced at the beginning by cap product with ξ^i's. $\square$

CHAPTER 4

CHERN CLASSES FOR HIGHER ALGEBRAIC K-THEORY WITH COEFFICIENTS IN THE MILNOR K-SHEAF

To generalise cohomology theories in algebraic geometry Gillet relies on analogies with topology. In this chapter we recall the axioms necessary for a generalised duality theory and apply this to the Milnor K-sheaf.

4.1 Chern classes with coefficients in a generalised cohomology theory

We recall Gillet's definitions and results [Gil81] for higher Chern classes with coefficients in a generalised cohomology theory. In short, the idea is that a cohomology theory with certain properties allows for the construction of universal classes over the classifying space $B.\,\mathbf{GL}_{n}.$, which in turn yield compatible universal classes for $\mathbf{GL}_n$. Using the Dold–Puppe functor, one finally obtains the desired characteristic classes. This is explained in more detail below.

Definition 4.1. *A graded cohomology theory Γ^* on a category of schemes $\mathscr{V}$ is a graded complex of sheaves of abelian group*

$$\underline{\Gamma}^*(*) = \bigoplus_{i\in\mathbb{Z}} \underline{\Gamma}^*(i)$$

on the big Zariski site of $\mathscr{V}$ together with an associative and graded-commutative pairing with unit in the derived category of graded complexes of abelian sheaves

$$\underline{\Gamma}^*(*) \otimes^L_{\mathbb{Z}} \underline{\Gamma}^*(*) \to \underline{\Gamma}^*(*).$$

In that way one may think of it being endued with a ring structure as we will discuss more closely later.

Definition 4.2. *Such a cohomology theory given, one may for a pair (Y, X), where Y is a closed subscheme of X, define the cohomology of X with support in Y by*

$$\mathrm{H}^i_Y(X, \Gamma(j)) = \mathbb{H}^i_Y(X, \underline{\Gamma}^*(j)).$$

Here $\mathbb{H}_Y$ denotes hypercohomology with supports.

This construction yields contravariant functors on the set of pairs (Y, X). A morphism $f : Z \to X$ induces natural maps for all i, j

$$f^! : \mathrm{H}^i_Y(X, \Gamma(j)) \to \mathrm{H}^i_{f^{-1}(Y)}(Z, \Gamma(j)).$$

The product on the graded complex $\underline{\Gamma}^*(*)$ induces a commutative ring structure on the associated cohomology $\mathrm{H}^*(X, \Gamma(*))$ and a $\mathrm{H}^*(X, \Gamma(*))$-module structure on $\mathrm{H}^*_Y(X, \Gamma(*))$. These structures are compatible with base changes via maps $f^!$.

Definition 4.3. *The definition is extended to simplicial objects in $\mathscr{V}$: If $X.$ is a simplicial scheme, for each $j \geqslant 0$, the complex $\underline{\Gamma}^*(j)$ of sheaves on $\mathscr{V}_{ZAR}$ restricts to a complex on $X.$, and $\mathrm{H}^i(X., \Gamma(j))$ are its hypercohomology groups.*

Examples 4.4. *Almost all known cohomology theories qualify as examples for this, although in many cases the grading does not play an important role or is constant. It does play a role in the theory of Chow groups associated to cycle modules as introduced in the previous section.*

It is desirable for many applications to have additional data and structures. This naturally leads to the definition of duality theories.

Definition 4.5. *Let $\mathscr{V}$ be a category of schemes over a fixed base S. A twisted duality theory on $\mathscr{V}$ with coefficients in a cohomology theory $\Gamma(*)$ consists of the following data (the number $d = 1, 2$ that appears in several axioms depends on the cohomology theory $\Gamma(*)$):*

1. ***Homology functor.*** *A covariant functor from $\mathscr{V}$ with proper morphisms into the category of bigraded abelian groups*

$$X \to \bigoplus_{\substack{i \geqslant 0 \\ j \in \mathbb{Z}}} \mathrm{H}_i(X, \Gamma(j)),$$

which is a Zariski presheaf for each X such that a Cartesian diagram with f, g proper and i, i' open immersions

$$\begin{tikzcd} U \arrow[r, "i'"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "i"] & Y \end{tikzcd}$$

leads to a commutative diagram in the target category

$$\begin{tikzcd} \mathrm{H}_i(U, \Gamma(j)) \arrow[d, "g_!"'] & \mathrm{H}_i(X, \Gamma(j)) \arrow[l, "i'^*"'] \arrow[d, "f_!"] \\ \mathrm{H}_i(V, \Gamma(j)) & \mathrm{H}_i(Y, \Gamma(j)) \arrow[l, "i^*"'] \end{tikzcd}$$

2. ***Localisation sequence.*** *Let (Y, i, X, j, U) be a boundary triple as mentioned earlier. Then there is a long exact sequence of homology*

$$\cdots \to \mathrm{H}_i(Y, \Gamma(j)) \xrightarrow{i_!} \mathrm{H}_i(X, \Gamma(j)) \xrightarrow{j^*} \mathrm{H}_i(U, \Gamma(j)) \xrightarrow{\partial} \mathrm{H}_{i-1}(Y, \Gamma(j)) \to \cdots$$

functorial in the way that for all proper morphisms $f : X \to X'$ there is a map of long exact sequences from the one associated with (Y, X) to the one associated with $(f(Y), X')$.

3. ***Cap product.*** *For each pair (Y, X) as in (2) a cap product*

$$\bigcap : \mathrm{H}_i(X, \Gamma(r)) \otimes \mathrm{H}^j_Y(X, \Gamma(s)) \to \mathrm{H}_{i-j}(Y, \Gamma(r-s)),$$

which is a pairing of presheaves on each X in $\mathscr{V}$ and such that for a Cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f_Y"'] & X \arrow[d, "f_X"] \\ Y' \arrow[r] & X' \end{tikzcd}$$

with f proper and Y' a closed subscheme of X', there is a projection formula

$$f_!(j) \cap z = f_!(j \cap f^!(z))$$

for $\alpha \in \mathrm{H}_i(X, \Gamma(r))$ and $z \in \mathrm{H}^j_Y(X', \Gamma(s))$.

4. ***Fundamental class.*** *Let $X \in \mathscr{V}$ be flat over S of relative dimension $\leqslant n$. There is a global section $\eta_X \in \mathrm{H}_{dn}(X, \Gamma(n))$. Here d is one or two depending on Γ.*

5. ***Poincaré duality.*** *Let $X \in \mathscr{V}$ be smooth over S of relative dimension n and $Y \to X$ a closed immersion. Then*
$$\eta_X \cap : \mathrm{H}_Y^{dn-i}(X, \Gamma(n-r)) \to \mathrm{H}_i(Y, \Gamma(r))$$
is an isomorphism, and the class of η_X in $\mathrm{H}_d\, n(X, \Gamma(n)) \cong \mathrm{H}^0(X, \Gamma(0))$ corresponds to the unit in the ring structure of $\mathrm{H}^*(X, \Gamma(*))$. If $Y \in \mathscr{V}$ is a subscheme of a smooth scheme X over S, then we can compute $\mathrm{H}_i(Y, \Gamma(r))$ as $\mathrm{H}_Y^{dn-i}(X, \Gamma(n-r))$.

6. ***Sections with support.*** *If $j : Y \to X$ is a closed immersion of smooth schemes over S, then the isomorphism*
$$\mathrm{H}^i(Y, \Gamma(r)) \cong \mathrm{H}_Y^{i+dp}(X, \Gamma(r+p))$$
induced by (5) is induced by a map
$$j_! : \underline{\Gamma}_Y^*(r) \to Rj^! \underline{\Gamma}_X^*(r+p)\,[dp]\,,$$
where $j^!$ is the functor "sections with support in Y."

7. ***Projection formula.*** *Let $j : Y \to X$ be a closed immersion of codimension p, smooth over S. Then the projection formula induced by the one from (3) via the duality (5)*
$$j_!(z) \cup a = j_!(z \cup j^!(a)),$$
for $z \in \mathrm{H}^p(Y, \Gamma(r))$ and $a \in \mathrm{H}^i(X, \Gamma(s))$, is represented by a commutative diagram in the derived category of complexes of sheaves of abelian groups
$$\begin{tikzcd}
Rj_! \underline{\Gamma}_Y^*(r) \otimes_{\mathbb{Z}}^L \underline{\Gamma}_X^*(s) \arrow[r, "1 \otimes j^!"] \arrow[d, "j_! \otimes 1"', "\sim"] & Rj_! \left(\underline{\Gamma}_Y^*(r) \otimes_{\mathbb{Z}}^L \underline{\Gamma}_Y^*(s) \right) \arrow[d] \\
Rj^! \underline{\Gamma}_X^*(r+p)[dp] \otimes_{\mathbb{Z}}^L \underline{\Gamma}_X^*(s) \arrow[dr] & Rj_! \left(\underline{\Gamma}_Y^*(r+s) \right) \arrow[d, "\sim"] \\
 & Rj^! \left(\underline{\Gamma}_X^*(s+r+p)\,[dp] \right)
\end{tikzcd}$$

8. ***Cross product.*** *Let $X, Y \in \mathscr{V}$ be quasi-projective over S. There are external products*

$$\boxtimes : \mathrm{H}_i(X, \Gamma(r)) \otimes \mathrm{H}_j(Y, \Gamma(s)) \to \mathrm{H}_{i+j}(X \times Y, \Gamma(r+s))$$

coming from the natural product

$$\mathrm{H}^*_X(M\Gamma(*)) \otimes_{\mathbb{Z}} \mathrm{H}^*_Y(N, \Gamma(*)) \to \mathrm{H}^*_{X \times Y}(M \times N, \Gamma(*)),$$

where $X \to M$, $X \to N$ are embeddings into smooth schemes.

9. ***Homotopy invariance.*** *For any $X \in \mathscr{V}$ and $p : \mathbb{A}^1_X \to X$ the natural map, the induced morphism*

$$p^* : \mathrm{H}^i(X, \Gamma(r)) \to \mathrm{H}^i(\mathbb{A}^1_X, \Gamma(r))$$

is an isomorphism.

10. ***Projective bundle formula.*** *For $n \in \mathbb{N}$, $X \in \mathscr{V}$ the natural product*

$$\mathrm{H}^*(\mathbb{P}^n_S, \Gamma(*)) \bigotimes_{\mathbb{Z}} \mathrm{H}_*(X, \Gamma(*)) \to \mathrm{H}_*(\mathbb{P}^n_X, \Gamma(*))$$

is surjective and for the natural projection $\pi : \mathbb{P}^n_X \to X$ and $\xi \in \mathrm{H}^d(\mathbb{P}^n_X, \Gamma(1))$ the inverse image of the hyperplane class at infinity by the map $\mathbb{P}^n_X \to \mathbb{P}^n_S$, there is an isomorphism

$$\sum_{p=0}^{n} \pi^*(\cdot) \cap \xi^p : \bigoplus_{p=0}^{n} \mathrm{H}_{i-dp}(X, \Gamma(rp)) \xrightarrow{\sim} \mathrm{H}_i(\mathbb{P}^n_X, \Gamma(r)).$$

11. ***Cycle class map.*** *A natural transformation of contravariant functors*

$$cycle : \mathrm{Pic}(\cdot) \to \mathrm{H}^d(\cdot, \Gamma(1)),$$

which extends to the cycle class map for effective Cartier divisors.

We will mention a few examples without going into details.

Examples 4.6. *Many of the standard examples fall under this definition.*

- *De Rham cohomology of varieties over a field k of characteristic 0, which have smooth embeddings. Set $\underline{\Gamma}^*_X(j) = \Omega^\bullet_{X/k}$ for $j \in \mathbb{N}_0$ and $= 0$ otherwise. Here $d = 2$.*

– *The Chow ring for varieties of finite type over a field. Set* $\underline{\Gamma}^*_X(j) = \mathscr{K}_j$*, the Quillen K-sheaf for* $j \in \mathbb{N}_0$ *and* $= 0$ *otherwise. Here* $d = 1$*. Note: this is a special case of the generalised Chow groups for cycle modules defined by Rost.*
– *Etale cohomology of schemes over* $\operatorname{Spec}(\mathbb{Z}\left[\frac{1}{n}\right])$*. Set* $\underline{\Gamma}^*(i) = Ru^*\mu_n(i)$*, where* u *is the functor from the étale to the Zariski site. Here* $d = 2$*.*

Recall now the definition of a theory of Chern classes for a category of schemes $\mathscr{V}$ and a cohomology theory $\Gamma(*)$.

Definition 4.7. *A theory of Chern classes on* $\mathscr{V}$ *with coefficients in* Γ *for representations of sheaves of groups on* $\mathscr{V}$ *assigns for each* $X \in \mathscr{V}$ *to any representation*

$$\rho : \mathscr{G} \to \mathbf{GL}(\mathscr{F})$$

on a locally free $\mathscr{O}_X$*-module* $\mathscr{F}$ *classes*

$$C_i(\rho) \in \mathrm{H}^{di}(X, \mathscr{G}, \Gamma(i)),$$

where d *is the same constant as in the definition of duality theory. We have an associated total class*

$$C.(\rho) = \prod C_i(\rho) \in \prod_{i \in \mathbb{N}_0} \mathrm{H}^{di}(X, \mathscr{G}, \Gamma(i)),$$

which is an element of the units of the cohomology ring $\mathrm{H}^*(X, \mathscr{G}, \Gamma(*))$*. These classes satisfy the following axioms.*

1. ***Functoriality.*** *Let* $f : X \to Y$ *be a morphism in* $\mathscr{V}$ *and* $\rho : \mathscr{G} \to \mathbf{GL}(\mathscr{F})$ *a representation of sheaves of groups in* Y *and* $\varphi : \mathscr{H} \to f^*\mathscr{G}$ *a homomorphism of sheaves of groups on* X*. Moreover, let*

$$f^*(\rho) \circ \varphi : \mathscr{H} \to \mathbf{GL}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{O}_X)$$

be the induced representation on X*. Then*

$$C.(f^*(\rho) \circ \varphi) = \varphi^*(f^*(C.(\rho))),$$

where

$$\varphi^* : \mathrm{H}^*(X, f^*\mathscr{G}, \Gamma(*)) \to \mathrm{H}^*(X, \mathscr{H}, \Gamma(*))$$

is the map induced by φ and

$$f^* : \mathrm{H}^*(Y, \mathscr{G}, \Gamma(i)) \to \mathrm{H}^*(X, f^*\mathscr{G}, \Gamma(i))$$

is induced by the natural transformation of functors

$$f^* : \{\mathscr{G}\text{-invariant sections of } \underline{\Gamma}^*_Y(i)\} \to \{f^*\mathscr{G}\text{-invariant sections of } \underline{\Gamma}^*_X(i)\}\,.$$

2. ***Whitney sum formula or additivity.*** *Let*

$$0 \to (\rho', \mathscr{F}') \to (\rho, \mathscr{F}) \to (\rho'', \mathscr{F}'') \to 0$$

be an exact sequence of representations of $\mathscr{G}$, then

$$\mathrm{C}.(\rho) = \mathrm{C}.(\rho') \cdot \mathrm{C}.(\rho'').$$

3. ***Tensor products.*** *Let $(\rho_1, \mathscr{F}_1)$ and $(\rho_2, \mathscr{F}_2)$ be representations of $\mathscr{G}$ and $(\rho_1 \otimes \rho_2, \mathscr{F}_1 \otimes \mathscr{F}_2)$ their tensor product, then*

$$\widetilde{\mathrm{C}}.(\rho_1 \otimes \rho_2) = \widetilde{\mathrm{C}}.(\rho_1) \circledast \mathrm{C}.(\rho_2),$$

where $\circledast$ is the product defined by the universal polynomials of Grothendieck and $\widetilde{\mathrm{C}}.$ is the augmented total Chern class

$$(\mathrm{rank}(\rho), \mathrm{C}.(\rho)) \in \widetilde{\mathrm{H}}^*(X, \mathscr{G}, \Gamma(*)) = \mathbb{Z} \times \left(\prod_{i \in \mathbb{N}_0} \mathrm{H}^{di}(X, \mathscr{G}, \Gamma(i))\right)^*.$$

4. ***Stability.*** *Let $\varepsilon : \{e\} \to \mathbf{GL}(\mathscr{O}_X) \cong \mathscr{O}_X^*$ be the trivial rank one representation. Then*

$$\mathrm{C}.(\varepsilon) = 1$$

the identity element in the cohomology ring.

5. ***Normalisation.*** *For any representation $\rho : \mathscr{G} \to \mathbf{GL}(\mathscr{F})$ the zero class is trivial*

$$C_0(\rho) = 1.$$

Gillet shows in [Gil81] that this definition is not void. We sketch his argumentation.

Theorem 4.8. *Let $\mathscr{V}$, S, Γ be as before, in particular, Γ satisfies the axioms of Definitions 4.1 and 4.5. Then there is a theory of Chern classes over $\mathscr{V}$ with coefficients in Γ satisfying the axioms of Definition 4.7.*

Proof. (Sketch) This is done in three steps.

The first step is to construct universal classes

$$C_{in} \in \mathrm{H}^{di}(B.\,\mathbf{GL}_n\,/S, \Gamma(i)),$$

for each $n \in \mathbb{N}_0$ where $B.\,\mathbf{GL}_n$ is the simplicial classifying space of the groups scheme $\mathbf{GL}_n$.

Let $E^n_.$ be the standard universal rank n vector bundle over $B.\,\mathbf{GL}_n$ and $\mathbb{P}(E^n_.)$ the associated projective bundle. Let $\xi \in \mathrm{H}^1(\mathbb{P}(E^n_.)$ be the tautological divisor defined by a Čech cochain on a hypercover in the usual way. Via the cycle class map, which was Axiom 4.5(11), ξ defines a class in $\mathrm{H}^d(\mathbb{P}(E^n_., \Gamma(1))$, which by abuse of notation is also denoted by ξ.

Using the projective bundle formula 4.5(10), one shows that for a rank n vector bundle $F.$ over a simplicial scheme $X.$ in the category $\mathscr{V}$, there is a natural isomorphism

$$\mathrm{H}^r(\mathbb{P}(F.), \Gamma(s)) \cong \bigoplus_{i=0}^{n-1} \mathrm{H}^{r-di}(X., \Gamma(s-i))$$

induced by multiplication with ξ. The proof uses the projective bundle formula 4.5 (10) required in the definition of a duality theory. In particular, this is true if we take $X. = B.\,\mathbf{GL}_n$ and $F. = E^n_.$. As a consequence of this decomposition it is possible to define the universal rank n Chern classes by the equation

$$\xi^n + p^*(C_1) \cup \xi^{n-1} + \cdots + p^*(C_n) = 0$$

in $\mathrm{H}^*(\mathbb{P}(E^n_.), \Gamma(*))$, where $p : \mathbb{P}(E^n_.) \to B.\,\mathbf{GL}_n$ is the natural projection.

The second step is to pass from the classes

$$C_i \in \mathrm{H}^{di}(B.\,\mathbf{GL}_n\,/S, \Gamma(i))$$

to classes

$$C_i \in \mathrm{H}^{di}(X, \mathbf{GL}_n(\mathscr{O}_X), \Gamma(i))$$

for every scheme X/S in $\mathscr{V}$. Viewing $\Gamma(i)$ as a complex of (injective) sheaves over the big Zariski site of S, we can compute the cohomology of the simplicial scheme $B.\,\mathbf{GL}_n$ as the cohomology of the total complex associated to the double complex

$$\oplus_{k,l} \operatorname{Hom}_{(\tilde{S})_{\mathrm{ZAR}}}(B_k\,\mathbf{GL}_n, \Gamma^l(i)).$$

On S_{ZAR} we have pairings of sheaves

$$\mathscr{H}om(X, B_k\,\mathbf{GL}_n) \times \mathscr{H}om(B_k\,\mathbf{GL}_n, \Gamma^l(i)) \to \mathscr{H}om(X, \Gamma^l(i)),$$

which induces a morphism

$$\begin{aligned} \operatorname{Hom}_{(\tilde{S})_{\mathrm{ZAR}}}(B_k\,\mathbf{GL}_n, \Gamma^l(i)) &\to \operatorname{Hom}_{(\tilde{S})_{\mathrm{ZAR}}}(\mathscr{H}om(X, B_k\,\mathbf{GL}_n), \Gamma^l(i)) \\ &\cong \operatorname{Hom}_{X_{\mathrm{ZAR}}}(B_k\,\mathbf{GL}_n(\mathscr{O}_X), \Gamma^l(i)), \end{aligned}$$

where the second isomorphism is the canonical one. As this is a morphism of double complexes, one can compute the cohomology of the associated total complex on each side to obtain a map

$$\mathrm{H}^{di}(B.\,\mathbf{GL}_n, \Gamma(i)) \to \mathrm{H}^{di}(X, \mathbf{GL}_n(\mathscr{O}_X), \Gamma(i)).$$

Hence the images of the $C_i \in \mathrm{H}^{di}(B.\,\mathbf{GL}_n\,/S, \Gamma(i))$ under this morphism yield the desired universal classes, which by abuse of notation we denote by the same symbol.

The third step is to define Chern classes for any representation $\rho : \mathscr{G} \to \mathscr{A}ut(\mathscr{F})$ of a sheaf of groups on a scheme X/S with $\mathscr{F}$ locally free of rank n. For such a representation determines a map in the homotopy category associated to the category of simplicial sheaves over X

$$[\rho] : B.\mathscr{G} \to B.\,\mathbf{GL}_n(\mathscr{O}_X)$$

and therefore by functoriality a morphism

$$[\rho]^* : \mathrm{H}^{di}(X, \mathbf{GL}_n(\mathscr{O}_X), \Gamma(i)) \to \mathrm{H}^{di}(X, \mathscr{G}, \Gamma(i)),$$

we can define

$$C_i(\rho) = [\rho]^*(C_i),$$

where the C_i on the right-hand side was defined in the previous paragraph. One checks now that this definition satisfies the properties of Chern classes given in

Definition 4.7, i.e., Functoriality, Whitney Sum Formula, Tensor Product and Stability. □

Let now $Y \subset X$ be a closed subscheme and let $K_i^Y(X)$ be K-theory with supports. We want to construct characteristic classes

$$C_{ij}^Y : K_j^Y(X) \to \mathrm{H}_Y^{di-j}(X, \Gamma(i)).$$

Later we will only need the case $Y = X$.

Applying the above theorem to the regular representation $\iota_n : \mathbf{GL}_n(\mathscr{O}_X) \to \mathrm{Aut}(\mathscr{O}_X^n)$, one obtains for each n characteristic classes $C_i(\iota_n) \in \mathrm{H}^{di}, \mathbf{GL}_n(\mathscr{O}_X), \Gamma(i))$, which are stable. Thus they have the property that for $m \geqslant n$, if we denote $i_{nm} : \mathbf{GL}_n(\mathscr{O}_X) \to \mathbf{GL}_m(\mathscr{O}_X)$ the natural injection, we have

$$i_{nm}^*(C_i(\iota_m)) = C_i(\iota_n),$$

where the left-hand side is over $\mathbf{GL}_m$ and the right-hand side over $\mathbf{GL}_n$. We note as well that the cohomology of $\mathbf{GL}_n(\mathscr{O}_X)$ is stable in the sense that for each $k \in \mathbb{N}_0$ there is m_k such that for all $m \geqslant m_k$ and for sheaves of abelian groups $\mathscr{M}$ on X on which $\mathbf{GL}_n(\mathscr{O}_X)$ and $\mathbf{GL}(\mathscr{O}_X)$ act trivially

$$\mathrm{H}^k(X, \mathbf{GL}_m(\mathscr{O}_X), \mathscr{M}) \cong \mathrm{H}^k(X, \mathbf{GL}(\mathscr{O}_X), \mathscr{M}).$$

Hence we obtain classes

$$C_i \in \mathrm{H}^{di}(X, \mathbf{GL}(\mathscr{O}_X), \Gamma(i)).$$

By a theorem of Brown and Gersten that relates the cohomology with respect to a group to the cohomology with respect to the associated classifying simplicial group via the Dold–Puppe construction $\mathscr{K}$, we have $\mathrm{H}^{di}(X, \mathbf{GL}(\mathscr{O}_X), \Gamma(i)) = \mathrm{H}^0(X, B.\,\mathbf{GL}(\mathscr{O}_X), \mathscr{K}(di, \Gamma(i)))$. Therefore we can consider C_i as a map of simplicial sheaves

$$C_i : B.\,\mathbf{GL}(\mathscr{O}_X) \to \mathscr{K}(di, \Gamma(i)).$$

Now one uses Quillen's trick. First apply Bousfield-Kan's integral completion functor to C_i, which yields a diagram

$$\begin{tikzcd} B.\,\mathbf{GL}(\mathscr{O}_X) \arrow[r, "C_i"] \arrow[d, "\phi_0"] & \mathscr{K}(di,\Gamma(i)) \arrow[d, "\phi_1"] \\ \mathbb{Z}_\infty\, B.\,\mathbf{GL}(\mathscr{O}_X) \arrow[r, "\mathbb{Z}_\infty C_i"] & \mathbb{Z}_\infty \mathscr{K}(di,\Gamma(i)) \end{tikzcd}$$

where ϕ_1 and ϕ_0 are induced by a natural transformation $\phi : \mathrm{id} \to \mathbb{Z}_\infty$. In particular ϕ_1 is a weak homotopy equivalence, so that we have an isomorphism of cohomology groups $\mathrm{H}^k_Y(X, \mathscr{K}(di,\Gamma(i))) \cong \mathrm{H}^k_Y(X, \mathbb{Z}_\infty \mathscr{K}(di,\Gamma(i)))$. The left hand side is related to the cohomology of $\Gamma(i)$ via an isomorphism

$$\mathrm{H}^{-j}_Y(X, \mathscr{K}(di,\Gamma(i))) \cong \mathrm{H}^{di-j}_Y(X,\Gamma(i)).$$

Moreover, we have a natural homomorphism for $j \in \mathbb{N}_0$

$$K^Y_j(X) \to \mathrm{H}^{-j}_Y(X, \mathbb{Z}\times\mathbb{Z}_\infty(B.\,\mathbf{GL}(\mathscr{O}_X))).$$

As the $\mathbb{Z}$-factor of $\mathbb{Z}\times\mathbb{Z}_\infty(B.\,\mathbf{GL}(\mathscr{O}_X))$ does not affect the Chern classes, we can put these maps together

$$\begin{tikzcd} K^Y_j(X) \arrow[r] \arrow[dddrr, "c^Y_{ij}"'] & \mathrm{H}^{-j}_Y(X,\mathbb{Z}\times\mathbb{Z}_\infty(B.\,\mathbf{GL}(\mathscr{O}_X))) \arrow[r] & \mathrm{H}^{-j}_Y(X,\mathbb{Z}_\infty(B.\,\mathbf{GL}(\mathscr{O}_X))) \arrow[d, "\mathrm{H}^{-j}(C_i)"] \\ & & \mathrm{H}^{-j}_Y(X,\mathbb{Z}_\infty\mathscr{K}(di,\Gamma(i))) \arrow[d, "\sim"] \\ & & \mathrm{H}^{-j}_Y(X,\mathscr{K}(di,\Gamma(i))) \arrow[d, "\sim"] \\ & & \mathrm{H}^{di-j}_Y(X,\Gamma(i)) \end{tikzcd}$$

This shows the existence of the Chern classes claimed.

Theorem 4.9. *Let $\mathscr{V}$ be a category of schemes over a fixed base S and $\Gamma(*)$ a duality theory satisfying the axioms of Definitions 4.1 and 4.5. Then there exists a theory of Chern classes for higher algebraic K-theory*

$$c_{ij} : K_j(X) \to \mathrm{H}^{di-j}(X,\Gamma(i))$$

satisfying the axioms of Definition 4.7 for representations of sheaves of groups in $\mathscr{V}$ with coefficients in Γ.

They are compatible with base change: let $f: X \to Y$ be a morphism of schemes in $\mathscr{V}$ and $Z \in Y$ a closed subscheme, then the diagram

$$\begin{tikzcd}
K_j^Z(Y) \arrow[r, "c_{ij}^Z"] \arrow[d, "f^*"] & \mathrm{H}^{di-p}(Y, \Gamma(i)) \arrow[d, "f^!"] \\
K_j^{f^{-1}(Z)}(X) \arrow[r, "c_{ij}^{f^{-1}(Z)}"] & \mathrm{H}^{di-p}_{f^{-1}(Z)}(X, \Gamma(i))
\end{tikzcd}$$

commutes. This follows from the axiom 4.7(1) of functoriality for a theory of Chern classes. They are also compatible with sums in the sense that for $\alpha, \beta \in K_j^Y(X)$ with $j > 0$ the formula

$$c_{ij}^Y(\alpha + \beta) = c_{ij}^Y(\alpha) + c_{ij}^Y(\beta)$$

holds. This is a consequence of the Whitney sum formula 4.7(2). Moreover, for the universal Chern classes $C_i \in \mathrm{H}^{di}(X, \mathbf{GL}(\mathscr{O}_X), \Gamma(i))$ and $C_j \in \mathrm{H}^{dj}(X, \mathbf{GL}(\mathscr{O}_X), \Gamma(j))$, the induced homomorphisms by their cup product

$$K_p^Y(X) \to \mathrm{H}_Y^{d(i+j)-p}(X, \Gamma(i+j))$$

is trivial for $p > 0$. The proof of the last two properties uses homotopy invariance 4.5(9) of the duality theory Γ.

4.2 Higher Chern classes for the Milnor K-sheaf

It was mentioned earlier that the Chow ring associated to Quillen's K-sheaves, which is a generalised cohomology theory in the sense of Definition 4.5, is a special case of Rost's Chow rings [Ros96]. Therefore, it makes sense to investigate if other cycle modules give rise to generalised cohomology theories as well. In particular we want to do this for the Milnor K-sheaves.

Let X be smooth over a field k of dimension n. Now set

$$\underline{\Gamma}_X^*(j) = \mathscr{K}_j^M$$

for $j \geqslant 0$ and the zero sheaf otherwise, where this is seen as a complex with only one spot nonzero, and further let $S = k$. By the Gersten Conjecture (Corollary 2.10)

$$\mathscr{K}_*^M = \begin{cases} \overline{\mathscr{K}}_*^M & \text{in the case of infinite residue fields} \\ \widehat{\mathscr{K}}_*^M & \text{in the case of finite residue fields.} \end{cases}$$

The associated generalised cohomology theory is

$$\mathrm{H}^i(X, \Gamma(j)) = \mathrm{H}^i(C^*(X; K_*^M, j)) = A^i(X; K_*^M, j).$$

Theorem 4.10. *There is a theory of Chern classes for vector bundles and higher algebraic K-theory of regular varieties over k with infinite residue fields, with values in Zariski cohomology with coefficients in the Milnor K-sheaf:*

$$c_{ij}^M : K_j(X) \to \mathrm{H}^{i-j}(X, \mathscr{K}_i^M).$$

Proof. We have to verify that the duality theory associated with $\underline{\Gamma}_X^*(j) = C^*(X; K_*^M, j)$ satisfies certain axioms of Definition 4.5, which are needed in the construction according to Theorems 4.8 and 4.9. At this point we only check the necessary axioms; for a discussion of the other ones see the appendix. Most of the properties are general properties of cycle modules and the associated Chow groups.

1. **Cap product.** Recall that there is a pairing of cycle modules

   $$K_*^M \times K_*^M \to K_*^M,$$

   which respects grading. Using the map "multiplication with units" from point 3 in Subsection 3.2, this induces a pairing of complexes

   $$C_p(X, K_*^M, j) \times C^q(X\big|_Y, K_*^m, i) \to C_{p-q}(Y, K_*^M, j-i),$$

   where we have used that $C^q = C_{n-q}$ as X is of dimension n and where C_Y^q means sections with support in Y. This map respects the grading on K_*^M since the original pairing on K_*^M does so. Moreover, it respects the grading in dimension as it is a generalised correspondence map mentioned in [Ros96, (3.9)]. Applying the (co)homology functor, we obtain a pairing

   $$\bigcap : A_p(X; \mathrm{K}_*^M, j) \otimes A_Y^q(X; K_*^M, i) \to A_{p-q}(Y; K_*^M, j-i).$$

2. **Projective bundle formula.** We proved this in Proposition 3.19

3. **Cycle class map.** This is clear from the definition of the first Milnor K-group. Indeed, recall that by definition of the Milnor K-sheaf

$$\mathscr{K}_1^M = \mathscr{O}_X^*,$$

and the well known isomorphism for a scheme X

$$\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathscr{O}_X^*)$$

gives a natural transformation of contravariant functors on the big Zariski site $\mathscr{V}$.

Now we can apply Theorem 4.8 and Theorem 4.9 and go through the construction sketched earlier to obtain the claim. $\square$

Remark 4.10.1. As we discuss in Section 2.4 this translates one-to-one to the étale case, and we can replace in the theorem Zariski cohomology with étale cohomology, with which we want to work.

CHAPTER 5

LOGARITHMIC WITT DIFFERENTIALS

Let k be a perfect field of characteristic $p > 0$. For a smooth k-scheme X let $W^{\dagger}\Omega_X$ be the overconvergent de Rham–Witt complex as defined by Davis, Langer and Zink in [DLZ11] and $W\Omega_X$ the usual de Rham–Witt complex [Ill79]. The goal of this section is to find a good notion of logarithmic overconvergent differentials and prove that they factor through Milnor K-theory.

5.1 Definition

For $n \in \mathbb{N}$ denote

$$d\log : \mathscr{O}_X^* \to W_n\Omega_X^1$$

the morphism of abelian sheaves defined locally by $x \mapsto \frac{d[x]}{[x]}$. This induces a morphism of projective systems

$$d\log : \mathscr{O}_X^* \to W_\bullet\Omega_X^1.$$

Let $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ be the subsheaf generated étale-locally by sections of the form $d\log[x_1]\ldots d\log[x_i]$ for $x_j \in \mathscr{O}_X^*$. This construction is known to be functorial in X, and the product structure of $W_n\Omega^\bullet_X$ carries over to $W_n\Omega^\bullet_{X,\log}$. For n variable, $W_\bullet\Omega^\bullet_{X,\log}$ is an abelian subprosheaf of $W_\bullet\Omega^\bullet_X$, and we set $W\Omega^\bullet_{X,\log} := \varprojlim W_\bullet\Omega^\bullet_{X,\log}$. For $i \in \mathbb{N}_0$ there is a short exact sequence of prosystems for étale topology

$$0 \to W_\bullet\Omega^i_{X,\log} \to W_\bullet\Omega^i_X \xrightarrow{\mathrm{F}-1} W_\bullet\Omega^i_X \to 0,$$

where F denotes a lift of the Frobenius endomorphism.

Taking the limit yields an exact sequence

$$0 \to W\Omega^i_{X,\log} \to W\Omega^i_X \xrightarrow{\mathrm{F}-1} W\Omega^i_X.$$

This means that $W\Omega^\bullet_{X,\log} = \mathrm{Ker}(\mathrm{F}-1) \subset W\Omega^\bullet_X$. It is not a priori clear that exactness holds also on the right; that is, that $\mathrm{F}-1$ is surjective.

Let $R_{nm} : W_n\Omega^\bullet_X \to W_m\Omega^\bullet_X$ be the restriction map for $n \geqslant m$. We want to show that for i fixed the projective system $(W_n\Omega^i_{X,\log}, R_{nm}$ satisfies the Mittag-Leffler condition locally. Indeed, since étale locally $W_n\Omega^i_{X,\log}$ is generated by sections of the form $d\log[x_1]_n \dots d\log[x_i]_n$, where the Teichmüller lifts are in truncated Witt vectors of length n, and the restriction maps R_{nm} commute with multiplication, addition and differential, we have for $n \geqslant m \geqslant k$

$$R_{nk}\left(\frac{d\,[x]_n}{[x]_n}\right) = R_{mk}\left(\frac{d\,[x]_m}{[x]_m}\right),$$

and thus locally

$$R_{nk}\left(W_n\Omega^\bullet_{X,\log}\right) = R_{mk}\left(W_m\Omega^\bullet_{X,\log}\right).$$

This induces exactness of the sequence

$$0 \to W\Omega^i_{X,\log} \to W\Omega^i_X \xrightarrow{F-1} W\Omega^i_X \to 0$$

for étale topology (but not for global sections).

5.2 Basic Witt differentials

Assume now that X is the spectrum of a polynomial algebra $A = k[X_1, \dots, X_d]$. Langer and Zink proved [LZ04, Theorem 2.8] that any element $\omega \in W\Omega^\bullet_A$ has a unique expression as a convergent sum of basic Witt differentials

$$\sum_{k,\mathscr{P}} e(\xi_{k,\mathscr{P}}, k, \mathscr{P}),$$

where k runs over all possible weight functions and $\mathscr{P}$ over all partitions of $\operatorname{Supp} k$ and for any m $\xi_{k,\mathscr{P}} \in^{V^m} W(k)$ for almost all weights k. The last condition is another way of saying that the sum converges p-adically.

A weight function is a function $k : [1,d] \to \mathbb{N}_0\left[\frac{1}{p}\right]$. The value of k at i is denoted by k_i. We call k integral if all values of k are integral. If k is integral, we set

$$X^k = X_1^{k_1} \cdots X_d^{k_d}.$$

Let $\operatorname{Supp} k$ be the support of k, i.e., the elements of the domain, where k does not vanish and fix for each k a total order on $\operatorname{Supp} k = \{i_1, \dots, i_r\}$ respecting the p-divisibility

$$\operatorname{ord}_p k_{i_1} \leqslant \cdots \leqslant \operatorname{ord}_p k_{i_r}.$$

To simplify notation, set $t(k) = -\operatorname{ord}_p k$ and $u(k) = \max(0, t(k))$. Think of $u(k)$ as the denominator of k. Let $I = \{i_\ell, i_{\ell+1}, \ldots, i_{\ell+m}\}$ be an interval of $\operatorname{Supp} k$ in the given order. If k_I is the restriction of k to I, set

$$\begin{aligned} t(k_I) &= t(k_{i_\ell}) = \max\{t(k_i) \mid i \in I\} \\ u(k_I) &= u(k_{i_\ell}) = \max(0, t(k_I)). \end{aligned}$$

For fixed k, let $\mathscr{P}$ be a partition of $\operatorname{Supp} k = I_0 \sqcup I_1 \sqcup \ldots \sqcup I_\ell$ respecting the order. The interval I_0 may be empty, but the intervals $I_1, \ldots, I_\ell$ not. A basic Witt differential $e = e(\xi, k, \mathscr{P}) \in W\Omega^\ell_A$ of degree ℓ with $\xi =^{\mathrm{V}^{u(I)}} \eta \in^{\mathrm{V}^{u(I)}} W(k)$ is defined in the following way: Denote by $r \in [0, \ell-1]$ the first index such that $k_{I_{r+1}}$ is integral. Three cases occur.

1. $I_0 \neq \varnothing$, no condition on the integrality of k.

$$\begin{aligned} e \;=\; & {}^{\mathrm{V}^{u(I_0)}}\left(\eta\, [X]^{p^{u(I_0)} k_{I_0}}\right) \left(d\,{}^{\mathrm{V}^{u(I_1)}}\, [X]^{p^{u(I_1)} k_{I_1}}\right) \cdots \left(d\,{}^{\mathrm{V}^{u(I_r)}}\, [X]^{p^{u(I_r)} k_{I_r}}\right) \cdot \\ & \left({}^{\mathrm{F}^{-t(I_{r+1})}} d\, [X]^{p^{t(I_{r+1})} k_{I_{r+1}}}\right) \cdots \left({}^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}\right) \end{aligned}$$

   Here $\xi =^{\mathrm{V}^{u(I_0)}} \eta$.

2. $I_0 = \varnothing$ and k not integral.

$$\begin{aligned} e \;=\; & \left(d\,{}^{\mathrm{V}^{u(I_1)}}\left(\eta\, [X]^{p^{u(I_1)} k_{I_1}}\right)\right) \cdots \left(d\,{}^{\mathrm{V}^{u(I_r)}}\, [X]^{p^{u(I_r)} k_{I_r}}\right) \cdot \\ & \left({}^{\mathrm{F}^{-t(I_{r+1})}} d\, [X]^{p^{t(I_{r+1})} k_{I_{r+1}}}\right) \cdots \left({}^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}\right) \end{aligned}$$

   Similarly as before $\xi =^{\mathrm{V}^{u(I_0)}} \eta$.

3. $I_0 = \varnothing$ and k integral.

$$e = \eta \left({}^{\mathrm{F}^{-t(I_1)}} d\, [X]^{p^{t(I_1)} k_{I_1}}\right) \cdots \left({}^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}\right).$$

   Here $\xi = \eta$.

If in $e(\xi, k, \mathscr{P})$ the element ξ is contained in ${}^{\mathrm{V}^m} W(R)$, then the image of the basic Witt differential under the restriction map R_m is trivial.

Proposition 5.1. *The action of* F, V *and* $\alpha \in W(k)$ *on the basic Witt differentials are given as follows:*

1. *$\alpha e(\xi, k, \mathscr{P}) = e(\alpha\xi, k, \mathscr{P})$.*

2. *If $I_0 \neq \emptyset$, or if k is integral (first and third case above),*

$$^{\mathrm{F}}e(\xi, k, \mathscr{P}) = e(^{\mathrm{F}}\xi, pk, \mathscr{P}).$$

3. *If $I_0 = \emptyset$ and k is not integral (second case above),*

$$^{\mathrm{F}}e(\xi, k, \mathscr{P}) = e(^{\mathrm{V}^{-1}}\xi, pk, \mathscr{P}).$$

4. *If $I_0 \neq \emptyset$ or k is integral and divisible by p,*

$$^{\mathrm{V}}e(\xi, k, \mathscr{P}) = e(^{\mathrm{V}}\xi, \frac{1}{p}k, \mathscr{P}).$$

5. *If $I_0 = \emptyset$ and $\frac{1}{p}k$ is not integral,*

$$^{\mathrm{V}}e(\xi, k, \mathscr{P}) = e(p^{\mathrm{V}}\xi, \frac{1}{p}k, \mathscr{P}).$$

This is Proposition 2.5 in [LZ04]. Note that if $\omega \in W\Omega_A$ is given as a unique decomposition in basic Witt differentials $\sum e$, then its image under Frobenius has the unique decomposition $\mathrm{F}\,\omega = \sum \mathrm{F}\,e$. In this sense one could say that the decomposition remains "fixed" under Frobenius. The types of basic Witt differentials are almost stable under the action of Frobenius, i.e., one could switch from type 2 to type 3, since the weight is multiplied by p, but this is the only switch from one type to another that can possibly occur. What is more, the Frobenius action is injective on the set of basic Witt differentials.

5.3 The overconvergent de Rham–Witt complex

Let A be a polynomial algebra over k. We recall the definition of the overconvergent de Rham–Witt complex [DLZ11]. Let $\omega = \sum_{k,\mathscr{P}} e(\xi, k, \mathscr{P}) \in W\Omega_A$ given as its unique decomposition as a sum of basic Witt differentials. For $\varepsilon > 0$ the Gauß norm is defined by

$$\gamma_\varepsilon(\omega) = \inf_{k,\mathscr{P}} \{\mathrm{ord}_V\, \xi_{k,\mathscr{P}} - \varepsilon|k|\}.$$

Definition 5.2. *An element $\omega = \sum_{k,\mathscr{P}} e(\xi, k, \mathscr{P}) \in W\Omega_A$ is said to be overconvergent of radius ε, if $\gamma_\varepsilon(\omega) > -\infty$.*

Note that the Teichmüller lift of an element in A is by default overconvergent.

Denote by $W^{\varepsilon}\Omega_A$ the overconvergent Witt differentials of radius ε. The overconvergent de Rham–Witt complex is the union over all possible constants $\varepsilon > 0$

$$\bigcup_{\varepsilon} W^{\varepsilon}\Omega_A =: W^{\dagger}\Omega_A,$$

which is a subdifferential graded algebra of $W\Omega_A$.

If $A = k[t_1, \dots, t_r]$ is a smooth finitely generated k-algebra and S the polynomial algebra $k[X_1, \dots, X_r]$, then the morphism $S \to A$, $X_i \mapsto t_i$ induces a canonical epimorphism

$$\lambda : W\Omega_S \to W\Omega_A$$

of de Rham–Witt complexes.

Definition 5.3. *We set* $W^{\dagger}\Omega_A = \lambda\left(W^{\dagger}\Omega_S\right)$.

This does not depend on the presentation, although the radii of overconvergence do in general.

Davis, Langer and Zink show that this construction can be globalised to a smooth scheme for étale and Zariski topology ([DLZ11, Cor. 1.7 and Thm. 1.8]). Thus for a (smooth) scheme X we have a subcomplex of the classical de Rham–Witt complex

$$W^{\dagger}\Omega_X \subset W\Omega_X.$$

Remark 5.3.1. Notice that by definition Witt differentials of finite length are overconvergent for **some** constant ε. Hence the natural morphism

$$W^{\dagger}\Omega_X \otimes \mathbb{Z}/p^n\mathbb{Z} \to W_n\Omega_X$$

is an isomorphism in the derived category of $\mathbb{Z}/p^n\mathbb{Z}$-modules. Indeed,

$$W^{\dagger}\Omega_X^i \otimes \mathbb{Z}/p^n\mathbb{Z} \to W_n^i\Omega_X$$

is evidently an isomorphism for all $i \geqslant 0$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules. On the other hand, the Tor-functor,

$$\mathrm{Tor}_j(\mathbb{Z}/p^n\mathbb{Z}, W^{\dagger}\Omega_X^i) = 0$$

for all $j > 0$ and all $i \geqslant 0$ as multiplication by p is injective in $W^\dagger\Omega^i_X$. Accordingly, there is an isomorphism

$$W\Omega_X \cong \varprojlim W^\dagger\Omega_X \otimes \mathbb{Z}\,/\,p^n\,\mathbb{Z}\,.$$

5.4 Log-differentials and the Steinberg relation

Let X be a smooth scheme over k. The map $d\log$ defined above induces a morphism

$$\begin{array}{rcl} d\log : \mathscr{O}_X^* & \to & W\Omega^1_{X,\log} \to W\Omega_X\,[1] \\ x & \mapsto & \dfrac{d\,[x]}{[x]} \end{array}$$

defined étale locally. The aforementioned fact that Teichmüller lifts are overconvergent along with the fact that the overconvergent complex is a subdifferential graded algebra of the classical de Rham–Witt complex shows that the elements $d\log\,[x_1]\ldots d\log\,[x_i]$ for $x_j \in \mathscr{O}_X^*$ are overconvergent. Therefore we have in fact a natural map

$$d\log : \mathscr{O}_X^* \to W\Omega^1_{X,\log} \to W^\dagger\Omega_X\,[1]\,.$$

Moreover, extending this to higher degrees yields for every $i \geqslant 0$ a morphism

$$\begin{array}{rcl} d\log^{\otimes i} : \mathscr{O}_X^* \otimes \cdots \otimes \mathscr{O}_X^* & \to & W\Omega^i_{X,\log} \to W^\dagger\Omega_X\,[i] \\ x_1 \otimes \cdots \otimes x_i & \mapsto & d\log(x_1)\cdots d\log(x_i). \end{array}$$

Definition 5.4. *We set $W^\dagger\Omega_{X,\log}$ to be the subcomplex of the overconvergent complex generated étale locally by logarithmic Witt differentials.*

Remark 5.4.1. In Section 6.4 we show the equality

$$W^\dagger\Omega_{X,\log} = W\Omega_{X,\log},$$

where the second complex is the logarithmic subcomplex of the normal de Rahm-Witt complex $W\Omega_X$, as both can be realised as the kernel of the map $1 - F$.

Remark 5.4.2. As Gros points out in [Gro85, Théorème 1.3.3], there is a natural isomorphism

$$W_\bullet\Omega_{X,\log} \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} W_n\Omega_{X,\log}$$

in the derived category of $\mathbb{Z}/p^n\mathbb{Z}$-promodules and therefore a natural isomorphism

$$W\Omega_{X,\log} \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} W_n\Omega_{X,\log}.$$

Moreover one has

$$W\Omega_{X,\log} \cong \varprojlim W\Omega^i_{X,\log} \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

In order to relate this to the Milnor K-sheaf, we prove the following:

Proposition 5.5. *The symbols $d\log(x_1)\cdots d\log(x_i)$ with $x_1,\ldots x_i \in \mathscr{O}_X^*$ satisfy the Steinberg relation.*

Proof. Let $x \in \mathscr{O}_X^*$ be a local section and assume that $1-x \in \mathscr{O}_X^*$. It is enough to show locally that $d\log(x)d\log(1-x) = 0$, or even that $d[x]d[1-x] = 0$.

Thus let $X = \operatorname{Spec} B$, where B is a quotient of a polynomial algebra over k. For an element $b \in B$ we calculate the expression $d[b]d[1-b]$. Consider the morphism of k-algebras

$$\psi : k[z] \to B$$

sending z to b. This morphism induces by functoriality a morphism of differential graded algebras between the associated de Rham–Witt complexes,

$$\psi : W\Omega_{k[z]} \to W\Omega_B,$$

which by abuse of notation we also denote by ψ. Yet the de Rham–Witt complex of $k[z]$ is trivial in degree greater than 1,

$$W\Omega_{k[z]} : 0 \to W\mathscr{O}_{k[z]} \to W\Omega^1_{k[z]} \to 0,$$

thence it is clear that $d[z]d[1-z]$ is zero. But as ψ is a morphism of differentially graded algebras, this implies already that $d[b]d[1-b]$ as the image in $W\Omega_B$ is zero as well.

From this it follows that in the general case for $x \in \mathscr{O}_X^*$

$$d\log(x) d\log(1-x) = \frac{d[x]}{[x]}\frac{d[1-x]}{[1-x]} = 0.$$

Using basic properties of the de Rham–Witt complex as differentially graded algebra, in particular anticommutativity, the claim follows. □

Example 5.6. *The argumentation above implies in particular that for X/k smooth and a local section $x \in \mathscr{O}_X$, we have the equality*

$$d[x]d^V[x] = 0.$$

Indeed, one can calculate $[1-x]$ using the Witt polynomials w_n and summation polynomials S_i to be

$$[1-x] = [1] - [x] - \sum_{i=1}^{\infty} V^i[S_i(1,0,\ldots;-x,0,\ldots)],$$

where the $S_i(1,0,\ldots;-x,0,\ldots)$ are sums of powers of x. Since we have seen in the proposition that $d[x]d[1-x] = 0$ and it is obvious that $d\,[1] = 0$ and $d[x]d[x] = 0$, it is clear that for any $i, j \in \mathbb{N}$

$$d[x]d^{V^i}[x^j] = 0$$

as well.

It is a good exercise to calculate this equality by hand. Consider for example the simplest case where $i = j = 1$. One has

$$\begin{aligned} ({}^V[x])d[x] &= {}^V\left([x]^{\mathrm{F}}d[x]\right) \\ &= {}^V\left([x][x]^{p-1}d[x]\right) \\ &= {}^V\left([x]^p d[x]\right) \\ &= {}^V\left([x]^p\right) d^V[x], \end{aligned}$$

where all equalities come from defining properties of F*-V procomplexes as mentioned in [LZ04, Section 1.2]. With this, one gets*

$$\begin{aligned} d^V[x]d[x] &= d\left({}^V[x]d[x]\right) \\ &= d\left({}^V\left([x]^p\right)d^V[x]\right) \\ &= d^V\left([x]^p\right)d^V[x] \\ &= {}^V\left([x]^{p-1}\right)d^V[x]d^V[x] = 0, \end{aligned}$$

where we used $d^V\left(\xi^p\right) = {}^V\left(\xi^{p-1}\right)d^V\xi$ for $\xi \in W(X)$ from [LZ04, Lemma 1.5].

Corollary 5.7. *Let X be a smooth scheme over k with infinite residue fields. For each i the map*

$$d\log^{\otimes i} : \mathscr{O}_X^* \otimes \cdots \otimes \mathscr{O}_X^* \to W^\dagger\Omega_X[i]$$

factor through $\mathscr{K}_i^M$ on X and $d\log^{\otimes i}$ can be augmented to a morphism of sheaves on X

$$d\log^i : \overline{\mathscr{K}}_i^M \to W^\dagger\Omega[i].$$

The next step is to show that the overconvergent de Rham–Witt complex is an object of $\mathfrak{ST}_{\text{ét}}$.

5.5 The transfer map for the overconvergent complex

It makes sense to view $W^\dagger\Omega$ as an abelian sheaf on the big étale site of all schemes.

We will now define a transfer for the overconvergent complex. Let $i : A \to B$ be a finite étale extension of local rings. Fix an explicit representation of the A-algebra $B \cong A[T]/(f)$ with $f \in A[T]$ monic such that f' is a unit. Let $G_{B/A} = \mathrm{Aut}_A(B)$ be the automorphism group of B that fixes A. Since B/A is étale, in particular unramified, and the extension is of finite degree, we know that

$$\#G_{B/A} = \deg(B/A).$$

By functoriality of the overconvergent complex, each element $g \in G_{B/A}$ induces a morphism of complexes

$$g : W^\dagger\Omega_B \to W^\dagger\Omega_B.$$

Lemma 5.8. *Let $g : B \to B$ be an automorphism that fixes A. Then the induced morphism of de Rham–Witt complexes is also an automorphism that fixes $W^{\dagger}\Omega_A$.*

Proof. By functoriality of the (overconvergent) de Rham–Witt complex, the induced morphism is an isomorphism. It fixes $W^{\dagger}\Omega_A$ because this is true for the usual de Rham–Witt complex by construction; thus the same holds true for the restriction to the overconvergent subcomplex. □

Lemma 5.9. *Let $\omega \in W^{\dagger}\Omega_B$ be fixed by all $g \in G_{B/A}$. Then ω is in fact in $W^{\dagger}\Omega_A$.*

Proof. Consider first the case when $x \in W(B)$. In this case, the elements of $G_{B/A}$ act componentwise, and the claim follows from the fact that A is exactly the fixed ring of $G_{B/A}$. This is of course also true if we restrict to the overconvergent Witt vectors.

To extend this results to the (overconvergent) de Rham–Witt complex, recall that there is an isomorphism

$$W\Omega_B \cong W(B) \otimes_{W(A)} W\Omega_A.$$

Compare the remark after Proposition 1.9 in [DLZ11]. It is clear that $W\Omega_A$ is fixed by the elements in $G_{B/A}$, so it comes down to the coefficients in $W(B)$, but for this ring we just showed the claim. Without difficulty this transfers over to the overconvergent subcomplex. □

Define the following map

$$\begin{aligned} N_{B/A} : W^{\dagger}\Omega_B &\to W^{\dagger}\Omega_B \\ \omega &\mapsto \sum_{g \in G_{B/A}} g\omega. \end{aligned}$$

Proposition 5.10. *The map $N_{B/A}$ has image in $W^{\dagger}\Omega_A$, and the restriction to $W^{\dagger}\Omega_A$ is multiplication by $\deg(B/A)$:*

$$N_{B/A} \circ i_* = \deg(B/A)\, \mathrm{id}_{W^{\dagger}\Omega_A}.$$

Proof. Let $h \in G_{B/A}$ and $\omega \in W^\dagger\Omega_B$. Then

$$\begin{aligned} hN_{B/A}(\omega) &= h \sum_{g \in G_{B/A}} g\omega \\ &= \sum_{g \in G_{B/A}} hg\omega \\ &= \sum_{g' \in G_{B/A}} g'\omega = N_{B/A}(\omega). \end{aligned}$$

Thus $N_{B/A}(\omega)$ is fixed by all elements of $G_{B/A}$ and by the previous lemma this means that $N_{B/A}(\omega) \in W^\dagger\Omega_A$.

For the second part of the claim, let $\omega \in W^\dagger\Omega_A$. Therefore, ω is fixed by $G_{B/A}$, and

$$N_{B/A}(\omega) = \sum_{g \in G_{B/A}} g\omega = (\#G_{B/A})\omega.$$

But we have seen that $\#G_{B/A} = \deg(B/A)$. This proves the claim. □

This shows that it makes sense to call the defined map $N_{B/A}$ a norm for the overconvergent complex for B over A.

Corollary 5.11. *Let $i : A \to B$ as before and A' a local A algebra such that $B' = B \otimes_A A'$ is also local. Let $i' : A' \to B'$ be the induced inclusion. Then the map $N_{B'/A'}$ is multiplicative with image in $W^\dagger\Omega_{A'}$ and*

$$N_{B'/A'} \circ i'_* = \deg(B/A)\,\mathrm{id}_{W^\dagger\Omega_{A'}}.$$

Proof. This follows directly as $\deg(B'/A') = \deg(B/A)$. □

Moreover we have the following compatibility.

Proposition 5.12. *Let $i : A \to B$ be as before. Let $f : A' \to A''$ be a morphism of local A-algebras such that $B' = B \otimes_A A'$ and $B'' = B \otimes_A A''$ are also local. Denote by $f^B : B' \to B''$ the induced morphism. Then the diagram*

$$\begin{tikzcd} W^\dagger\Omega_{B'} \arrow[r] \arrow[d, "N_{B'/A'}"'] & W^\dagger\Omega_{B''} \arrow[d, "N_{B''/A''}"] \\ W^\dagger\Omega_{A'} \arrow[r] & W^\dagger\Omega_{A''} \end{tikzcd}$$

commutes.

Proof. The ring extensions B'/A' and B''/A'' are both finite étale of degree $\deg(B/A)$ since B/A is finite étale and all are local rings. What is more, the corresponding automorphism groups are isomorphic

$$G_{B/A} \cong G_{B'/A'} \cong G_{B''/A''}.$$

Thus we see that for $\omega \in W^{\dagger}\Omega_{B'}$

$$\begin{aligned} f_* \circ N_{B'/A'}(\omega) &= f_* \sum_{g' \in G_{B'/A'}} g'\omega \\ &= \sum_{g' \in G_{B'/A'}} f_* g'\omega \\ &= \sum_{g'' \in G_{B''/A''}} g'' f_*^B \omega \\ &= N_{B''/A''} \circ f_*^B(\omega). \end{aligned}$$

Due to functoriality of $W^{\dagger}\Omega$ and by a similar statement for rings. □

Comparing what we have just shown with the definition of the category $\mathfrak{S}\mathfrak{T}_{\text{ét}}$ in Section 2.3 yields indeed the desired result.

Corollary 5.13. *The sheaf $W^{\dagger}\Omega$ is an object of $\mathfrak{S}\mathfrak{T}_{\text{ét}}$.*

5.6 Continuity of the overconvergent complex

Continuity of a functor means that it commutes with filtering direct limits. The de Rham–Witt complex does not commute with general direct limits and is thus not continuous. Although the overconvergent one might be, we content ourselves to show that it commutes with direct limits of finite k-algebras. Since in our context only smooth schemes over k appear, which are in particular locally of finite type, this is enough.

Lemma 5.14. *Let*

$$A = \varinjlim A_i$$

be a filtering direct limit in the category of finite k-algebras. Then for the functor of Witt vectors the natural homomorphism

$$\varinjlim W(A_i) \to W(A)$$

is an isomorphism.

Proof. Consider the ghost maps

$$w_i : W(A_i) \to A_i^{\mathbb{N}} \qquad \text{and} \qquad w : W(A) \to A.$$

These are by definition ring homomorphisms. Because of the fact that the ring structure on the image of the ghost map is defined componentwise, the natural map

$$\varinjlim (A_i^{\mathbb{N}}) \to \left(\varinjlim A_i\right)^{\mathbb{N}} = A^{\mathbb{N}}$$

is a monomorphism. Since we have only finitely many generators it is in fact an isomorphism. By definition of the Witt vectors the following diagram commutes

$$\begin{tikzcd}
\varinjlim W(A_i) \arrow[r] \arrow[d, "w"'] & W(A) \arrow[d] \\
\varinjlim (A_i^{\mathbb{N}}) \arrow[r, "\sim"] & A^{\mathbb{N}}
\end{tikzcd}$$

and the vertical maps are injective. Additionally, we have just seen that the bottom line is an isomorphism. Thus it is clear that the top line is a monomorphism. Now take an element $a \in W(A)$. To see that it has a pre-image in $\varinjlim W(A_i)$, project it down to $A^{\mathbb{N}}$ via the ghost map. The image $w(a)$ has a pre-image $\widetilde{w(a)} \in \varinjlim (A_i^{\mathbb{N}})$. However, as the element $w(a)$ comes from a Witt vector given in Witt components, it is possible by solving the corresponding equations recursively to recover an element $\tilde{a} \in \varinjlim W(A_i)$ that maps to a. □

Proposition 5.15. *The de Rham–Witt complex is continuous on the category of finite k-algebras.*

Proof. We have to show that for a filtering direct limit of finite k-algebras

$$A = \varinjlim A_i$$

the natural map

$$\varinjlim W\Omega_{A_i} \to W\Omega_{\varinjlim A_i}$$

is an isomorphism. We will show that $W\Omega_A$ satisfies the universal property of direct limits in the category of Witt complexes over A. Let M be a Witt complex over A. This means that it is a differentially graded $W(A)$ algebra with morphisms F and V that satisfy certain properties. Keeping in mind that by functoriality there is a morphism $W(A_i) \to W(A)$, we see that M is also a Witt complex over each A_i. Therefore it makes sense to consider maps

$$W\Omega_{A_i} \to M.$$

In fact, for each i there is exactly one map of this form because the de Rham–Witt complex is the initial object in the category of Witt complexes over A_i. The same is true for the de Rham–Witt complex over A in the category of Witt complexes over A. Thus there is exactly one map

$$W\Omega_A \to M.$$

What is more, this map is compatible with the ones over A_i, i.e., for each i the diagram

$$\begin{tikzcd} W\Omega_{A_i} \arrow[r] \arrow[d] & M \arrow[d] \\ W\Omega_A \arrow[r] & M \end{tikzcd}$$

where the upper horizontal morphism consists of $W(A_i)$-algebras and the bottom one of $W(A)$-algebras, commutes. This follows from the fact that we have an isomorphism $\varinjlim W(A_i) \xrightarrow{\sim} W(A)$ as shown in the last lemma. The claim follows. $\square$

Corollary 5.16. *The overconvergent de Rham–Witt complex is continuous on the category of finite k-algebras.*

Proof. Let $\varinjlim A_i = A$ be a filtered direct system of finite k-algebras. The restriction of the natural isomorphism

$$\varinjlim W\Omega_{A_i} \to W\Omega_A$$

to the overconvergent subcomplexes $W^\dagger\Omega_{A_i}$ has image in the overconvergent subcomplex $W^\dagger\Omega_A$. In fact it is exactly the natural homomorphism

$$\varinjlim W^\dagger\Omega_{A_i} \to W^\dagger\Omega_A$$

induced by functoriality from $\varinjlim A_i \xrightarrow{\sim} A$ and the diagram

$$\begin{tikzcd}
\varinjlim W^\dagger\Omega_{A_i} \arrow[r] \arrow[d] & W^\dagger\Omega_A \arrow[d] \\
\varinjlim W\Omega_{A_i} \arrow[r] & W\Omega_A
\end{tikzcd}$$

commutes.

The morphism on the overconvergent complex is injective as the original map on the whole de Rham–Witt complex is injective. To check surjectivity, assume that $\omega \in W^\dagger\Omega_A$ with radius $\varepsilon > 0$. By the previous proposition there is a unique preimage $\widetilde{\omega} \in \varinjlim W\Omega_{A_i}$. In particular, for each i, there is a $\omega_i \in W\Omega_{A_i}$ which maps to ω, and it has to be overconvergent. Let $\varepsilon_i > 0$ be its radius of overconvergence. Up to choosing different presentations, we may assume without loss of generality that the radii ε_i are bounded below by ε. Hence, $\widetilde{\omega}$ can be considered as an element of $\varinjlim W^\dagger\Omega_{A_i}$. $\square$

5.7 The transformation map

Let us briefly collect the facts that we established throughout this section:

In the last two subsections we have shown that the complex $W^\dagger\Omega$ on the big étale site of all schemes is an object of the category $\mathfrak{ST}_{\text{ét}}$, continuous on finite k-algebras. One should note that this is sufficient for our case, although it is a priori a restriction of Kerz's definition because we only wish to apply our functors to such cases. In fact, the overconvergent complex as suggested in [DLZ11] is only defined for finite k-algebras.

Moreover, we have seen that $\overline{\mathscr{K}}^M_n$ is for every n a continuous object of in $\mathfrak{ST}^\infty_{\text{ét}}$ and that there exists a continuous $\widehat{\mathscr{K}}^M_n \in \mathfrak{ST}_{\text{ét}}$ and a natural transformation $\overline{\mathscr{K}}^M_n \to \widehat{\mathscr{K}}^M_n$ satisfying a universal property. This comprises all ingredients needed to apply Theorem 2.17.

As mentioned earlier, there is for each n a morphism of continuous étale sheaves

$$d\log^n : \overline{\mathscr{K}}^M_n \to W^\dagger\Omega\,[n]\,.$$

As a consequence of Theorem 2.17, which is based on Kerz's result (cf. Corollary 2.14), we obtain a unique natural transformation of étale sheaves

$$\widehat{d\log}^{n} : \widehat{\mathscr{K}}_{n}^{M} \to W^{\dagger}\Omega\,[n]\,. \tag{5.1}$$

For simplicity, we will use the notation

$$d\log^{n} : \mathscr{K}_{n}^{M} \to W^{\dagger}\Omega\,[n] \tag{5.2}$$

for the general case, where $\mathscr{K}_{n}^{M}$ is the sheaf $\overline{\mathscr{K}}_{n}^{M}$ in the infinite residue field case and $\widehat{\mathscr{K}}_{n}^{M}$ in the finite residue field case.

CHAPTER 6

CHERN CLASSES WITH COEFFICIENTS IN THE OVERCONVERGENT DE RHAM–WITT COMPLEX

In this chapter we relate the Milnor K-sheaf to the overconvergent de Rham–Witt complex in order to obtain overconvergent Chern classes.

6.1 Definition

Let X/k be a smooth variety. The map $d\log^n : \mathscr{K}_n^M \to W^\dagger\Omega[n]$ defined in the previous section induces a morphism on cohomology

$$\mathrm{H}^m(X, \mathscr{K}_i^M) \to \mathbb{H}^{m+i}(X, W^\dagger\Omega_X), \tag{6.1}$$

which by abuse of notation we denote as well by $d\log$.

It follows that the Chern classes $c_{ij}^M : K_j(X) \to \mathrm{H}^{i-j}(X, \mathscr{K}_i^M)$ from Theorem 4.10 induce Chern classes with coefficients in the overconvergent complex.

Theorem 6.1. *Let X be a smooth scheme over k. There is a theory of Chern classes for vector bundles and higher algebraic K-theory of regular varieties over k, with values with coefficients in the overconvergent de Rham–Witt complex:*

$$c_{ij}^{sc} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X).$$

Remark 6.1.1. Note that the maps $d\log^n$ respect the descending filtration of the de Rham–Witt complex by the differential graded ideals. Thus we obtain in fact Chern classes into the bigraded cohomology groups

$$c_{ij}^{sc} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^\dagger\Omega_X^{\geqslant i}).$$

We will now look into some properties of the Chern classes just defined.

6.2 Comparison to crystalline Chern classes

In this section assume that X/k is proper in addition to being smooth.

By construction the morphism $d\log^i : \mathscr{K}_i^M \to W^\dagger\Omega\,[i]$ factors for each i through $W^\dagger\Omega^i_{\log} = W\Omega^i_{\log}$. (For said equality see Section 6.4.) In the same manner as above, one may define logarithmic Chern classes

$$c_{ij}^{\log} : K_j(X) \to \mathbb{H}^{i-j}(X, W\Omega^i_{\log}),$$

which factor the overconvergent Chern classes. Thus it is natural to ask how these Chern classes compare to the logarithmic Chern classes with finite coefficients of Gros in [Gro85]

$$c_{ij}^{\mathrm{Gros}} : K_j(X) \to \mathbb{H}^{i-j}(X, W_n\Omega^i_{\log}).$$

Proposition 6.2. *The diagram*

$$\begin{tikzcd} & \mathrm{H}^{i-j}\left(X, W\Omega^i_{\log}\right) \arrow[dd] \\ K_j(X) \arrow[ur, "c_{ij}^{\log}"] \arrow[dr, "c_{ij}^{Gros}"'] & \\ & \mathrm{H}^{i-j}\left(X, W_n\Omega^i_{\log}\right) \end{tikzcd}$$

commutes.

Proof. Recall first Gros' construction. Let $\pi : \mathscr{E} \to X$ be a vector bundle of constant rank $r+1$ and $\mathbb{P}(\mathscr{E})$ the associated projective bundle. Gros uses essentially standard methods to build his logarithmic Chern classes by first constructing $c_1^{\mathrm{Gros}}(\mathscr{O}_{\mathbb{P}(E)})$ of the projective line bundle using differential logarithms in [Gro85, Section I.2], and then showing that a projective bundle formula

$$\mathbb{H}^{r+1}(\mathbb{P}(E), W_n\Omega^{r+1}_{\mathbb{P}(E),\log}) = \bigoplus_{i=0}^{r} \mathrm{H}^{r+1-i}(X, W_n\Omega^{r+1-i}_{X,\log}).c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^i$$

holds. Then the classes $c_i(E)$ for $0 \leqslant i \leqslant r+1$ are uniquely defined. We will take advantage of this fact to show that Gros' Chern classes factor through the Milnor Chern classes.

The differential logarithm map

$$d\log : \mathscr{O}_X^* \to W_n\Omega^1_{X,\log}$$

as described by Gros, induces a map on the Milnor K-sheaf

$$d\log^i : \mathscr{K}_i^M \to W_n\Omega^i_{X,\log}$$

in the same way as above, and we have a commutative diagram of sheaves

$$\begin{tikzcd} & W\Omega^i_{\log} \arrow[dd, "\otimes \mathbb{Z}/p^n\mathbb{Z}"] \\ \mathscr{K}_i^M \arrow[ur] \arrow[dr] & \\ & W_n\Omega^i_{\log} \end{tikzcd} \tag{6.2}$$

This map induces also morphisms of cohomology groups

$$\mathrm{H}^m(X, \mathscr{K}_i^M) \to \mathrm{H}^m(X, W_n\Omega^i_{\log}),$$

which is in particular compatible with multiplication in the cohomology ring.

The fact that in both cases—for the Milnor K-sheaf and for the logarithmic de Rham–Witt complex with finite coefficients—we have at our disposal a projective bundle formula limits the comparison problem to the first Chern class. Indeed, the diagram

$$\begin{tikzcd} & \mathrm{H}^1(X, \mathscr{K}_1^M) \arrow[dd, "d\log"] \\ \mathrm{H}^1(X, \mathscr{O}_X^*) \arrow[ur, "\mathrm{id}"] \arrow[dr, "d\log"'] & \\ & \mathrm{H}^1(X, W_n\Omega^1_{\log}) \end{tikzcd}$$

is commutative per definitionem and the upper morphism defines c_1^M, whereas the lower one defines c_1^{Gros} (cf. [Gro85, Section I.2]). Consequently, we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^{i-j}(X, \mathscr{K}_i^M) \arrow[dd] \\
K_j(X) \arrow[ur, "c_{ij}^M"] \arrow[dr, "c_{ij}^{\mathrm{Gros}}"'] & \\
 & \mathrm{H}^{ij}(X, W_n\Omega^i_{\log})
\end{tikzcd}$$

which by using (6.2) extends to a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^{i-j}(X, \mathscr{K}_i^M) \arrow[r, "d\log"] \arrow[dd] & \mathrm{H}^{i-j}(X, W\Omega^i_{\log}) \arrow[ddl, "\otimes \mathbb{Z}/p^n\mathbb{Z}"] \\
K_j(X) \arrow[ur, "c_{ij}^M"] \arrow[dr, "c_{ij}^{\mathrm{Gros}}"'] & & \\
 & \mathrm{H}^{i-j}(X, W_n\Omega^i_{\log}) &
\end{tikzcd}$$

and this is exactly what we claimed. $\square$

If the projective system of logarithmic differentials $\left(W_n\Omega^i_{\log}, R_{nm}, R_n\right)$, where $R_{nm} : W_n\Omega^i \to W_m\Omega^i$ for $n \geqslant m$ and $R_n : W\Omega^i \to W_n\Omega^i$ are the restriction maps, was strict, we could see that the natural morphism

$$\mathbb{H}^m(X, W\Omega^i_{\log}) \to \varprojlim \mathbb{H}^m(X, W_n\Omega^i_{\log})$$

was an isomorphism (see the argumentation of [Ill79, II, Prop. 2.1]). Then Gros' logarithmic Chern classes would induce Chern classes with $\mathbb{Z}_p$ coefficients, which would lead to overconvergent Chern classes without having to use Milnor K-sheaves and Gillet's machinery of generalised duality theories.

Because of properness of X as assumed in the beginning of this section, we may compare the overconvergent Chern classes to crystalline Chern classes. From the discussion above and Gros' results we see that the diagram

$$\begin{tikzcd}
\mathscr{O}_X \otimes \cdots \otimes \mathscr{O}_X \arrow[r] \arrow[drr] & \mathscr{K}_i^M \arrow[r] & W\Omega^i_{X,\log} \arrow[r] \arrow[d, "\otimes \mathbb{Z}/p^n\mathbb{Z}"] & W^\dagger\Omega_X\,[i] \arrow[d, "\otimes \mathbb{Z}/p^n\mathbb{Z}"] \\
 & & W_n\Omega^i_{X,\log} \arrow[r] & W_n\Omega_X\,[i]
\end{tikzcd}$$

commutes. Since Gros shows in [Gro85, Section 2.1] that the logarithmic Chern classes he defines factor the crystalline Chern classes with $\mod p^n$ coefficients, the commutativity of the above diagram shows that the overconvergent Chern classes do the same, and one obtains a commutative diagram

$$\begin{tikzcd} & \mathbb{H}^{2i-j}\left(X, W^{\dagger}\Omega\right) \arrow[dd] \\ K_j(X) \arrow[ur, "c_{ij}^{\mathrm{sc}}"] \arrow[dr, "c_{ij}^{\mathrm{cris},n}"'] & \\ & \mathbb{H}^{2i-j}\left(X, W_n\Omega\right) \end{tikzcd}$$

that compares the overconvergent classes with finite level crystalline Chern classes. By reason that this diagram holds for all levels and that $\mathrm{H}^i_{\mathrm{cris}}(X/W) = \mathbb{H}^i(X, W\Omega)$, which means in particular, that the cohomology functor and the inverse limit commute, we have in fact a commutative diagram

$$\begin{tikzcd} & \mathbb{H}^{2i-j}\left(X, W^{\dagger}\Omega\right) \arrow[dd] \\ K_j(X) \arrow[ur, "c_{ij}^{\mathrm{sc}}"] \arrow[dr, "c_{ij}^{\mathrm{cris}}"'] & \\ & \mathrm{H}^{2i-j}_{\mathrm{cris}}\left(X/W\right) \end{tikzcd}$$

6.3 Overconvergent Chern classes and the γ-filtration

6.3.1 The λ-structure on Quillen K-theory

It is well known that Quillen's K-theory groups have a λ-structure—more precisely for a given scheme X $K_0(X)$ is a λ-ring and by Soulé the groups $K_m(X)$ can be equipped with a $K_0(X)$-λ algebra structure. We recall briefly the mechanism [Sou85].

A λ-ring is a unitary commutative ring R together with maps $\lambda^k : R \to R$ for $k \in \mathbb{N}_0$ satisfying

$$\begin{aligned}
\lambda^0(x) &= 1\\
\lambda^1(x) &= x\\
\lambda^k(1) &= 0 \quad \text{for} \quad k \geqslant 2\\
\lambda^k(x+y) &= \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)\\
\lambda^k(xy) &= P_k(\lambda^1(x),\dots,\lambda^k(x);\lambda^1(y),\dots,\lambda^k(y))\\
\lambda^k(\lambda^l(x)) &= P_{kl}(\lambda^1(x),\dots,\lambda^{kl}(x)),
\end{aligned}$$

where P_k and P_{kl} are universal integral polynomials.

The Grothendieck group of representations of $\mathbf{GL}_N$ over $\mathbb{Z}$, which is denoted by $R_{\mathbb{Z}}(\mathbf{GL}_N)$, together with the exterior power operation is a λ-ring isomorphic to the algebra $\mathbb{Z}[\lambda^0(\mathrm{id}_N), \lambda^0(\mathrm{id}_N)^{-1}, \dots, \lambda^N(\mathrm{id}_N), \lambda^N(\mathrm{id}_N)^{-1}$, where $\lambda^k(\mathrm{id}_N)$ is the k^{th} exterior power of the natural representation.

Let A be a unitary commutative ring. There is a natural morphism of abelian groups

$$r_N : R_{\mathbb{Z}}(\mathbf{GL}_N) \to [B.\,\mathbf{GL}_N(A), B.\,\mathbf{GL}_N(A)^+],$$

where the right-hand side consists of homotopy classes of pointed maps between the arguments. Taking limits on both sides and taking into account that

$$[B.\,\mathbf{GL}_N(A), B.\,\mathbf{GL}_N(A)^+] \cong [B.\,\mathbf{GL}_N(A)^+, B.\,\mathbf{GL}_N(A)^+]$$

by the universal property of the plus construction, we have in fact a canonical morphism

$$r_A : R_{\mathbb{Z}}(\mathbf{GL}) \to [B.\,\mathbf{GL}(A)^+, B.\,\mathbf{GL}(A)^+].$$

We call natural operation of λ-rings the data for each λ-ring R of a map

$$\tau : R \to R$$

such that it commutes with morphisms of λ-rings. In particular the λ^k are natural operations of λ-rings. For an operation τ of λ-rings the elements $\tau(\mathrm{id}_N - N) \in R_{\mathbb{Z}}(\mathbf{GL}_N)$, for each N are compatible with the inclusions $\mathbf{GL}_N \to \mathbf{GL}_M$ for a natural number $M \geqslant N$, to the effect that they determine an element of $R_{\mathbb{Z}}(\mathbf{GL}_N)$ and therefore, an element

$$\tau_A \in [B.\,\mathbf{GL}(A)^+, B.\,\mathbf{GL}((A)^+].$$

As the Quillen K-theory groups are the homotopy groups of $B.\,\mathbf{GL}(A)^+$ one obtains by functoriality morphisms

$$\tau_A : K_m(A) = \pi_m(B.\,\mathbf{GL}(A)^+) \to K_m(A).$$

For $m = 0$ this gives back the definition of λ-structure for $K_0(A)$ given by Grothendieck. Globally we also have morphisms $\tau_X : K_m(X) \to K_m(X)$ because the construction glues. Thus the operations λ^k induce in particular morphisms

$$\lambda^k : K_m(X) \to K_m(X)$$

and make the ring $\oplus_{m \in \mathbb{N}_0} K_m(X)$ into a $K_0(X)$-λ-algebra.

Moreover, we have additional structure. One defines the operation γ^k as a shift of λ^k

$$\gamma^k(x) = \lambda^k(x + k - 1) \quad \text{for} \quad k \geqslant 0$$

and the Adams operations ψ^k recursively by

$$\psi^k - \lambda^1\psi^{k-1} + \ldots + (-1)^{k-1}\lambda^{k-1}\psi^1 + (-1)^k k\lambda^k = 0.$$

Let A be as above. Then $K(A) = \oplus K_m(A)$ admits an augmentation map

$$\varepsilon : K(A) \to \mathbb{Z}^{\pi_0(\operatorname{Spec} A)}$$

projecting $K(A)$ onto $K_0(A)$ and then associating to an A-module of finite type its rank over each connected component of $\operatorname{Spec} A$. This enable us to define a decreasing filtration on $K(A)$

$$\begin{aligned} F^0_\gamma K(A) &= K(A) \\ F^j_\gamma K(A) &= \langle \gamma^{i_1}(x_1) \cdots \gamma^{i_n}(x_n) \,\big|\, \varepsilon(x_1) = \cdots = \varepsilon(x_n) = 0,\, i_1 + \cdots + i_n \geqslant j \rangle. \end{aligned}$$

We denote by $gr^i_\gamma K(A) = F^i_\gamma K(A)/F^{i+1}_\gamma K(A)$ the corresponding grading. This grading has nice properties, for example for $K_0(A)$ it gives back the Chow groups up to torsion. An operation of λ-rings τ respects this γ-filtration and is moreover given on the graded pieces by a universal integral constant $\omega_i(\tau) \in \mathbb{Z}$. Some of these are

$$\begin{aligned}
\omega_i(\psi^k) &= k^i \\
\omega_i(\lambda^k) &= (-1)^{k-1}k^{i-1} \\
\omega_i(\gamma^k) &= \begin{cases} 0 & \text{if } k > i \\ (-1)^{i-1}(i-1)! & \text{if } k = i \\ \neq 0 & \text{if } k \leqslant i. \end{cases}
\end{aligned}$$

From the definitions above we see that the operation γ^k can be defined as the image of the element $\gamma^k(\mathrm{id}_N - N)$ in $R_{\mathbb{Z}}(\mathbf{GL})$.

6.3.2 Milnor Chern classes and the γ-filtration

Let X be smooth over k, no restrictions on the residue fields. Considering the fact that the overconvergent Chern classes are defined via Milnor K-theory, it suffices to study the behaviour of the classes

$$c_{ij}^M : K_j(X) \to \mathrm{H}^{i-j}(X, \mathscr{K}_i^M)$$

on the filtration.

As mentioned in Section 4.1 we know from [Gil81, Lemma 2.26] that the c_{ij}^M for $j > 0$ are group homomorphisms, which follows from the Whitney Sum Formula. In order to study how the Chern classes act on the γ-filtration, we take a look at the product structure on K-theory. The multiplication as described by Loday is induced by a map

$$\mu_0 : B.\,\mathbf{GL}(\mathscr{O}_X)^+ \times B.\,\mathbf{GL}(\mathscr{O}_X)^+ \to B.\,\mathbf{GL}(\mathscr{O}_X)^+.$$

Arguing as in [Gil81, Lemma 2.32] we see that there is a commutative diagram

$$\begin{tikzcd}
B.\,\mathbf{GL}(\mathscr{O}_X)^+ \wedge B.\,\mathbf{GL}(\mathscr{O}_X)^+ \arrow[r, "\mu_0"] \arrow[d, "C_{\cdot}^M \wedge C_{\cdot}^M"] & B.\,\mathbf{GL}(\mathscr{O}_X)^+ \arrow[d, "C_{\cdot}^M"] \\
\prod_{i\in\mathbb{N}} \mathscr{K}(di, \mathscr{K}_i^M) \wedge \prod_{i\in\mathbb{N}} \mathscr{K}(di, \mathscr{K}_i^M) \arrow[r, "*"] & \prod_{i\in\mathbb{N}} \mathscr{K}(di, \mathscr{K}_i^M)
\end{tikzcd} \tag{6.3}$$

where $*$ is Grothendieck's multiplication [G+71] and $C_{\cdot}^M$ is the total Chern class.

Lemma 6.3. *If $\alpha \in K_l(X)$ and $\alpha' \in K_q(X)$, then*

$$c_{ij}^M(\alpha\alpha') = -\sum_{r+s=i} \frac{(i-1)!}{(r-1)!(s-1)!} c_{rl}^M(\alpha) c_{sq}^M(\alpha'),$$

where $l+q=j$.

Proof. By property (3) of Definition 4.7 we know that for the tensor product of two representations

$$\widetilde{C}_{\cdot}^M(\rho_1 \otimes \rho_2) = \widetilde{C}_{\cdot}^M(\rho_1) * C_{\cdot}(\rho_2),$$

where $\widetilde{C}_{\cdot}^M$ is the total augmented Chern class, and the product is as above in the diagram the Grothendieck multiplication which after Shekhtman (see [Niz98, Section 2]) is given by universal polynomials

$$(\sum_{i \geqslant 1} x_i) * (\sum_{j \geqslant 1} y_i) = \sum_{l \geqslant 0} P_l(x_1, \ldots, x_l, y_1, \ldots, y_l)$$

with $P_l(x_1, \ldots, y_l)) \sum_{r+s=l} a_{rs} x_r y_s + Z_r(x) T_s(y)$. Here

$$a_{rs} = -\frac{(l-1)!)}{(r-1)!(s-1)!}$$

and Z_r and T_s are polynomials of weight r and s, respectively, and for $r+s=l$ at least one of them is decomposable. Explicitly, this means for the Loday multiplication μ_0

$$\mu_0^* C_{\cdot}^M = \sum_{l \geqslant 0} \left(\sum_{r+s=l} \left(a_{rs} p_1^* C_r^M \cdot p_2^* C_s^M + Z_r(p_1^* C_{\cdot}^M) T_s(p_2^* C_{\cdot}^M) \right) \right),$$

where $p_i : \mathbf{GL}(\mathscr{O}_X) \times \mathbf{GL}(\mathscr{O}_X) \to \mathbf{GL}(\mathscr{O}_X)$ are the natural projections. Thus we have to show that the terms $Z_r(p_1^* C_{\cdot}^M) T_s(p_2^* C_{\cdot}^M)$ disappear when evaluated on the corresponding K-theory classes. The argumentation is analogue to the proof of [Gil81, Lemma 2.25]. An element $\eta \in K_j(X)$, $j \geqslant 1$ is represented by a map $\eta : \mathscr{S}_X^j \to B_{\cdot}\,\mathbf{GL}(\mathscr{O}_X)^+$ in the homotopy category, where $\mathscr{S}_X^j$ is the simplicial version of the j-sphere. For any $a, b \in \mathbb{N}$, the class $(C_a^M \cdot C_b^M)(\eta)$ is represented by the commutative diagram

$$\begin{tikzcd}
\mathscr{S}^j_X \arrow[rr, "\Delta_{\mathscr{S}^j_X}"] \arrow[d, "\eta"] & & \mathscr{S}^j_X \wedge \mathscr{S}^j_X \arrow[d, "\eta\wedge\eta"] \\
B.\,\mathbf{GL}(\mathscr{O}_X)^+ \arrow[r, "\Delta"] \arrow[d, "C^M_a\cdot C^M_b"] & B.\,\mathbf{GL}(\mathscr{O}_X)^+ \wedge B.\,\mathbf{GL}(\mathscr{O}_X)^+ \arrow[r, phantom] & {} \arrow[d, "C^M_a\wedge C^M_b"] \\
\mathscr{K}(a+b, \mathscr{K}^M_{a+b}) & \mathscr{K}(a, \mathscr{K}^M_a)\wedge \mathscr{K}(b, \mathscr{K}^M_b) \arrow[l, "\mu_{a,b}"'] &
\end{tikzcd}$$

and as a consequence we have the equalities

$$(C^M_a \cdot C^M_b)\eta = \mu_{a,b}(C^M_a \wedge C^M_b)\Delta\eta = \mu_{a,b}(C^M_a \wedge C^M_b)(\eta\wedge\eta)\Delta_{\mathscr{S}^j_X}.$$

Since the map $\Delta_{\mathscr{S}^j_X}$ is null-homotopic, this composition of maps is null-homotopic as well. The decomposable part of the $Z_r T_s$ is made up by terms of this form; hence it disappears and with it the whole expression. □

Lemma 6.4. *The integral Chern class maps c^M_{ij} restrict to zero on $F^{i+1}_\gamma K_j(X)$ for $i \geqslant 1$.*

Proof. In light of the previous formula and the Whitney sum formula, it suffices to show that c^M_{ij} is trivial on elements of the form $\gamma_k(x)$ for $k \geqslant i+1$ and $x \in K_j(X)$. Recall from above that the operation γ_k on $K_j(X)$ is defined to be the image of $\gamma_k(\mathrm{id}_N - N)$ under the natural map

$$r : R_{\mathbb{Z}}(\mathbf{GL}) \to [B.\,\mathbf{GL}(\mathscr{O}_X)^+, B.\,\mathbf{GL}(\mathscr{O}_X)^+] \to \mathrm{Hom}(K_j(X), K_j(X)).$$

Following [Gil81, Definition 2.27] we define the augmented cohomology ring

$$\widetilde{\mathrm{H}}^*(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}^M_i)$$

to be

$$\mathrm{H}^0(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathbb{Z}) \times \{1\} \times \mathrm{H}^i(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}^M_i),$$

where $\mathrm{H}^i(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}^M_i) = [\mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}(i, \mathscr{K}^M_i)]$. The ring $\widetilde{\mathrm{H}}^*(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}^M_i)$ is a strict λ-ring [Gil81, 2.27], and the augmented Chern class map

$$\begin{aligned} \widetilde{C}^M : R_{\mathbb{Z}}(\mathbf{GL}) &\to \widetilde{\mathrm{H}}^*(X, \mathbb{Z}\times B.\,\mathbf{GL}(\mathscr{O}_X)^+, \mathscr{K}^M_i) \\ \rho &\mapsto \Big(\mathrm{rank}(\rho), C^M_{\cdot}(\rho)\Big) \end{aligned}$$

is a λ-ring homomorphism.

It suffices to show that the class of $c^M_{i,N}(\gamma_k(\mathrm{id}_N - N))) \in \mathrm{H}^i(X, \mathbf{GL}_N(\mathscr{O}_X), \mathscr{K}^M_i)$ is trivial for $k \geqslant i+1$. By the previous paragraph, the Chern polynomial $c^M_{t,N}$ is a λ-ring homomorphism as well, and hence commute with the γ-operation. With the usual formulae we get

$$\begin{aligned} c^M_{t,N}(\gamma_k(\mathrm{id}_N - N)) &= \gamma_k(c^M_{t,N}(\mathrm{id}_N - N)) \\ &= \gamma_k(1 + c^M_{1,N}(\mathrm{id}_N - N)t + \cdots) \\ &= 1 + (-1)^{k-1}(k-1)! c^M_{k,N}(\mathrm{id}_N - N)t^k + \cdots \end{aligned}$$

and therefore $c^M_{i,N}(\mathrm{id}_N - N)) = 0$ for $i < k$. □

Corollary 6.5. *If $\alpha \in F^j_\gamma K_l(X)$, $j \neq 0$ and $\alpha' \in F^k_\gamma K_q(X)$, then*

$$c^M_{j+k,l+q}(\alpha\alpha') = -\frac{(j+k-1)!}{(j-1)!(k-1)!} c^M_{jl}(\alpha) c^M_{kq}(\alpha').$$

6.3.3 Passage to overconvergent Chern classes

Because the map

$$\mathrm{H}^m(X, \mathscr{K}^M_i) \to \mathbb{H}^{m+i}(X, W^\dagger\Omega)$$

induced by the morphism of complexes $\mathscr{K}^M_i \to W^\dagger\Omega[i]$ is a morphism of cohomology rings and therefore respects the respective operations, the results from the previous section carry over to the overconvergent Chern classes. This is summarised in the following proposition.

Proposition 6.6. *Let X/k be smooth.*

1. *If $\alpha \in K_l(X)$ and $\alpha' \in K_q(X)$, then*

$$c^{sc}_{ij}(\alpha\alpha') = -\sum_{r+s=i} \frac{(i-1)!}{(r-1)!(s-1)!} c^{sc}_{rl}(\alpha) c^{sc}_{sq}(\alpha'),$$

   where $l + q = j$.

2. *The integral Chern class maps c^{sc}_{ij} restrict to zero on $F^{i+1}_\gamma K_j(X)$ for $i \geqslant 1$.*

3. *If $\alpha \in F^j_\gamma K_l(X)$, $j \neq 0$ and $\alpha' \in F^k_\gamma K_q(X)$, then*

$$c^{sc}_{j+k,l+q}(\alpha\alpha') = -\frac{(j+k-1)!}{(j-1)!(k-1)!} c^{sc}_{jl}(\alpha) c^{sc}_{kq}(\alpha').$$

6.4 The action of $1 - \mathrm{F}$ on the overconvergent de Rham–Witt complex

In this section, we want to adapt some short exact sequences from [Ill79] to the overconvergent context.

We generalise the notion of basic Witt differentials to the case when A is of the form $k\left[X_1, X_1^{-1}, \ldots, X_d, X_d^{-1}\right]$. See [Ill79] and the proof of Proposition 1.3 in [DLZ11]. A basic Witt differential $e \in W\Omega_A$ has one of the following shapes:

1. e is a classical basic Witt differential in variables $[X_1], \ldots, [X_d]$.
2. Let $J \subset \{1, \ldots, d\}$ be a subset and denote by $e(\xi, k, \mathscr{P}, J)$ a basic classical Witt differential in $\{X_j \,|\, j \in J\}$.
   (a) $e = e(\xi, k, \mathscr{P}, J) \prod_{j \notin J} d \log [X_j]$.
   (b) $e = \prod_{j \notin J} [X_j]^{-r_j} e(\xi, k, \mathscr{P}, J)$ for some $r_j \in \mathbb{N}$.
   (c) $e = \prod_{j \notin J} {}^{\mathrm{F}^{s_j}} d [X_j]^{-l_j} e(\xi, k, \mathscr{P}, J)$ for some $l_j \in \mathbb{N}$, $p \nmid l_j$, $s_j \in \mathbb{N}_0$.
3. $e = {}^{\mathrm{V}^u}\left(\xi \prod_{j \notin J} [X_j]^{p^u k_j} [X]^{p^u k_{I_0}}\right) d^{\mathrm{V}^{u(I_1)}} [X]^{p^{u(I_1)} k_{I_1}} \ldots {}^{\mathrm{F}^{-t(I_\ell)}} d [X]^{p^{t(I_\ell)} k_{I_\ell}}$. In particular, for each such e, there is a weight function on variables $\{X_j \,|\, j \in J\}$ with partition $\mathscr{P}$, $u > 0$, $k_{j \notin J} \in \mathbb{Z}_{<0}\left[\frac{1}{p}\right]$ and $u(k_j) \leqslant u = \max\{u(I_0), u(k_j)\}$.
4. $e = de'$ where e' as in (3).

6.4.1 The action of F, V and p on $W^\dagger\Omega$

Proposition 6.7. *The action of* F *on the generalised basic Witt differentials are given as follows:*

1. *If e is a classical basic Witt differential in variables $[X_1], \ldots, [X_d]$, the action is given as in Proposition 5.1.*
2. *Let $J \subset \{1, \ldots, d\}$ be a subset and denote by $e(\xi, k, \mathscr{P}, J)$ a basic classical Witt differential in $\{X_j \,|\, j \in J\}$.*
   (a) If $e = e(\xi, k, \mathscr{P}, J) \prod_{j \notin J} d \log [X_j]$, then

$$
{}^{\mathrm{F}}e = \left({}^{\mathrm{F}}e(\xi, k, \mathscr{P}, J)\right) \prod_{j \notin J} d \log [X_j] \,.
$$

(b) *If* $e = \prod_{j \notin J} \left[X_j\right]^{-r_j} e(\xi, k, \mathscr{P}, J)$ *for some* $r_j \in \mathbb{N}$*, then*

$$^{\mathrm{F}}e = \prod_{j \notin J} \left[X_j\right]^{-pr_j} \left(^{\mathrm{F}}e(\xi, k, \mathscr{P}, J)\right).$$

(c) *If* $e = \prod_{j \notin J} {}^{\mathrm{F}^{s_j}} d \left[X_j\right]^{-l_j} e(\xi, k, \mathscr{P}, J)$ *for some* $l_j \in \mathbb{N}$*,* $p \nmid l_j$*,* $s_j \in \mathbb{N}_0$*, then*

$$^{\mathrm{F}}e = \prod_{j \notin J} {}^{\mathrm{F}^{s_j+1}} d \left[X_j\right]^{-l_j} \left(^{\mathrm{F}}e(\xi, k, \mathscr{P}, J)\right).$$

3. *If* $e = {}^{\mathrm{V}^u}\left(\xi \prod_{j \notin J} \left[X_j\right]^{p^u k_j} [X]^{p^u k_{I_0}}\right) d^{\mathrm{V}^{u(I_1)}} [X]^{p^{u(I_1)} k_{I_1}} \ldots {}^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}$*, then*

$$^{\mathrm{F}}e = {}^{\mathrm{V}^u}\left({}^{\mathrm{F}}\xi \prod_{j \notin J} \left[X_j\right]^{p^u k'_j} [X]^{p^u k'_{I_0}}\right) d^{\mathrm{V}^{u(I_1)}} [X]^{p^{u(I_1)} k'_{I_1}} \ldots {}^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k'_{I_\ell}},$$

   where $k' = pk$*.*

4. *If* $e = de'$ *where* e' *as in (3), the expression changes similar to the previous case, with the only difference that we get* ${}^{\mathrm{V}^{-1}}\xi$ *instead of* ${}^{\mathrm{F}}\xi$*.*

Proof. This is a straightforward calculation, using the definition of Frobenius and Verschiebung. □

In particular, F has the same stabilizing properties on the types of generalised basic Witt differentials as mentioned at the end of Section 5.2 with respect to the usual basic Witt differentials.

Remark 6.7.1. In this concrete case we can give a criterion when an element $\omega = \sum e(\xi, k, \mathscr{P})$ of the de Rham–Witt complex given as its decomposition in basic generalised Witt differentials is overconvergent based on the proof of Proposition 1.3 in [DLZ11]. Namely, ω is overconvergent if there exist constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that the basic Witt differentials e appearing in the decomposition satisfy the following conditions:

– If e is of type (1) or of type (2.a),

$$|k| \leqslant C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2.$$

– If e is of type (2.b),

$$|r| + |k| \leqslant C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2,$$

where $|r| = \sum r_j$.

– If e is of type (2.c),

$$|l \cdot p^s| + |k| \leqslant C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2,$$

where $|l \cdot p^s| = \sum l_j \cdot p^{s_j}$.

– If e is of type (3) or (4),

$$\sum |k_j| + \sum |k_{I_i}| \leqslant C_1 \operatorname{ord}_p({}^{V^u}\xi) + C_2,$$

where $|k_j| = -k_j$ and $|k_{I_i}| = \sum_{m \in I_i} k_m$.

The multiplication of an element $\alpha \in W(k)$ on a generalised basic Witt differential is particularly easy to define.

Proposition 6.8. *The action of $\alpha \in W(k)$ on a generalised basic Witt differential is given by multiplying the coefficient ξ with α.*

Proof. Analogues to the discussion in [LZ04, p.40], we see that the coefficients in the generalised basic Witt differentials are elements of ${}^{\mathrm{V}^{u(I)}}W(k)$, where I is the partition of the support of the weight function in question and u is defined as in Section 5.2. As for any $\alpha \in W(k)$ and $\xi \in {}^{\mathrm{V}^{u(I)}}W(k)$ the product $\alpha\xi$ is again in ${}^{\mathrm{V}^{u(I)}}W(k)$, we see as in [LZ04, loc.sit.] that multiplying by α a general basic Witt differential of one of the forms given at the begin of the section means to multiply the coefficient ξ appearing there by α. $\square$

In particular, multiplication by an element in $W(k)$ respects the types of general basic Witt differentials, and this is the fact that we will use later for multiplication by p.

Due to the fact that the Verschiebung map is only additive, but not a ring homomorphism, its action on the generalised basic Witt differentials is more complicated to describe. We have recalled the action on the usual basic Witt differentials in Proposition 5.1. We note further that

$${}^{\mathrm{V}}d\log[X_i] = d^{\mathrm{V}}\log[X_i]$$

as the general formula ${}^{\mathrm{V}}(\omega_0 d\omega_1 \cdots d\omega_i) =^{\mathrm{V}} \omega_0 d^{\mathrm{V}}\omega_1 \cdots d^{\mathrm{V}}\omega_i$ holds. This formula also tells us that for differentials of type (3), i.e., if

$$e =^{\mathrm{V}^u} \left(\xi \prod_{j \notin J} [X_j]^{p^u k_j} [X]^{p^u k_{I_0}}\right) d^{\mathrm{V}^{u(I_1)}} [X]^{p^{u(I_1)} k_{I_1}} \ldots ^{\mathrm{F}^{-t(I_\ell)}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}},$$

then

$${}^{\mathrm{V}}e =^{\mathrm{V}^u+1} \left(\xi \prod_{j \notin J} [X_j]^{p^u k_j} [X]^{p^u k_{I_0}}\right) d^{\mathrm{V}^{u(I_1)+1}} [X]^{p^{u(I_1)} k_{I_1}} \ldots ^{\mathrm{F}^{-t(I_\ell)-1}} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}.$$

Furthermore, it is possible to describe the action of V on elements of type (4) using the same formula by multiplying with the factor 1, which allows us to write

$${}^{\mathrm{V}}de = {}^{V}1 d^{\mathrm{V}}e,$$

which changes the coefficient to $p^{\mathrm{V}}\xi$ (see [LZ04, p. 41]), and we obtain

$${}^{\mathrm{V}}e = d^{\mathrm{V}^u} \left(p^{\mathrm{V}}\xi \prod_{j \notin J} [X_j]^{p^u k_j} [X]^{p^u k_{I_0}}\right) d^{\mathrm{V}^{u(I_1)}+1} [X]^{p^{u(I_1)} k_{I_1}} \ldots ^{\mathrm{F}^{-t(I_\ell)}-1} d\, [X]^{p^{t(I_\ell)} k_{I_\ell}}.$$

As for general basic differentials of type (2), this depends on the different cases and also on the form of the basic Witt differentials $e(\xi, k, \mathscr{P}, J)$ involved.

6.4.2 The kernel and cokernel of $1 - \mathrm{F}$ for $X = \mathbb{G}_m^d$

The logarithmic differentials are by definition of the de Rham–Witt complex fixed by the Frobenius endomorphism. One would like to prove that these are the only elements with this property. We start by making the assertion for the special case, when X is a product of multiplicative groups, i.e., $X = \operatorname{Spec} A$ with $A = k\left[X_1, X_1^{-1}, \ldots, X_d, X_d^{-1}\right]$.

Proposition 6.9. *The sections of $W^{\dagger}\Omega_{X,\log}$ on $X = \mathbb{G}_m^d$ are exactly the Frobenius fixed elements of $W\Omega_X$.*

Proof. It is clear that $W^{\dagger}\Omega_{X,\log} \subset (W\Omega_X)^{\mathrm{F}-1}$. For the converse, let $\omega \in (W\Omega_X)^{\mathrm{F}-1}$, with decomposition in generalised basic Witt differentials $\omega = \sum e$. However, we know that the action of Frobenius preserves the types of basic Witt differentials and that the decomposition is unique. We want to use this to argue that it is enough to check the generalised basic Witt differentials.

With the assumption $\mathrm{F}\,\omega = \omega$ there are two cases to consider: finite and infinite sets of elements appearing in unique the sum decomposition of ω that form a (finite respectively infinite) "cycle" under the action of F.

The finite case: assume that among the basic Witt differentials in the decomposition $\omega = \sum e$ there is a finite set $e_1, \ldots, e_n$ such that

$$\mathrm{F} \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} e_i,$$

which means after possibly reordering

$$\mathrm{F}\,e_1 = e_2\ ,\ \mathrm{F}\,e_2 = e_3\ ,\ \ldots,\ \mathrm{F}\,e_n = e_1.$$

But according to Proposition 6.7 this is impossible unless $n = 1$.

The infinite case: assume that in the decomposition there is an infinite set $e_1, e_2, \ldots$ such that

$$\mathrm{F} \sum_{i=1}^{\infty} e_i = \sum_{i=1}^{\infty} e_i,$$

which means after reordering

$$\mathrm{F}\,e_1 = e_2\ ,\ \mathrm{F}\,e_2 = e_3\ ,\ \ldots$$

However, this is not convergent p-adically and therefore not feasible.

Thus if the whole sum $\omega = \sum e$ is fixed under Frobenius, every basic differential appearing in this sum must be so.

According to Proposition 6.7, the sections fixed under Frobenius action are of the form (2.a) with $e(\xi, k, \mathscr{P}, J)$ trivial. From this we conclude that $(W\Omega_X)^{\mathrm{F}-1} \subset W^\dagger\Omega_{X,\log}$. $\square$

It follows in particular that we have for all $i \in \mathbb{N}$ locally for étale topology a commutative diagram where the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & W\Omega^i_{X,\log} \arrow[r, hook] & W\Omega^i_X \arrow[r, "\mathrm{F}-1"] & W\Omega^i_X \arrow[r] & 0 \\
0 \arrow[r] & W^\dagger\Omega^i_{X,\log} \arrow[r, hook] \arrow[u, equal] \arrow[ur, hook] & W^\dagger\Omega^i_X \arrow[r, "\mathrm{F}-1"] \arrow[u, hook] & W^\dagger\Omega^i_X \arrow[u, hook] &
\end{tikzcd}$$

Earlier we mentioned the condition for an element $\omega \in W\Omega_{X/k}$ to be overconvergent. On the other hand, this shows that an element $\omega \in W\Omega_{X/k}$ is **not** overconvergent, if for all $C_1 > 0$ and $C_2 \in \mathbb{R}$ there is an elementary Witt differential e in its decomposition violating one of the inequalities. More precisely,

– If e is of type (1) or of type (2.a),

$$|k| > C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2.$$

– If e is of type (2.b),

$$|r| + |k| > C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2,$$

where $|r| = \sum r_j$.

– If e is of type (2.c),

$$|l \cdot p^s| + |k| > C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2,$$

where $|l \cdot p^s| = \sum l_j \cdot p^{s_j}$.

– If e is of type (3) or (4),

$$\sum |k_j| + \sum |k_{I_i}| > C_1 \operatorname{ord}_p({}^{V^u}\xi) + C_2,$$

where $|k_j| = -k_j$ and $|k_{I_i}| = \sum_{m \in I_i} k_m$.

We will study the action of Frobenius on elementary Witt differentials subject to the inequalities indicating overconvergence or nonoverconvergence.

As mentioned before, any Witt differential over $\mathbb{G}_m^d$ can be written in a unique way as a sum of basic Witt differentials. The basic Witt differentials appearing in this sum are characterised by a coefficient, a weight function and a partition. The action of Frobenius was described in Proposition 6.7. In essence Frobenius action changes the weights and the coefficients of the basic differentials, creating a new unique sum. One can see Frobenius as being injective on the **set** of basic Witt differentials.

We want to argue that overconvergence is preserved by Frobenius if we modify one of the constants in an obvious way.

Lemma 6.10. *Let e be a basic Witt differential satisfying an inequality indicating overconvergence for constants C_1 and C_2, then so does ${}^{F}e$ for constants C_1 and pC_2.*

Proof. This is shown for one type at a time.

– If e is of type (1) or of type (2.a), then the inequality depends only on the basic classical Witt differential appearing in the expression, namely

$$|k| \leqslant C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2.$$

The action of F on a differential of this type changes the weight k to pk and the coefficient $\xi_{k,\mathscr{P}}$ to ${}^{\mathrm{F}}\xi_{k,\mathscr{P}}$ if k is integral and ${}^{V^{-1}}\xi_{k,\mathscr{P}}$ if k is fractional; the partition is essentially unchanged. The inequality has to be modified to

$$|pk| = p|k| \leqslant C_1 \operatorname{ord}_p({}^{\mathrm{F}}\xi_{k,\mathscr{P}}) + pC_2.$$

– If e is of type (2.b), the crucial inequality is

$$|r| + |k| \leqslant C_1 \operatorname{ord}_p \xi_{k,\mathscr{P}} + C_2,$$

where $|r| = \sum r_j$. The action of F changes r_j to pr_j and therefore $|r|$ to $p|r|$; k and ξ change as in the previous case. As above, it becomes clear that the only modification to the constants has to be pC_2 instead of C_2.

– If e is of type (2.c), the same argument is valid with $|l|$ in the place of $|r|$.

– If e is of type (3) or (4), action of Frobenius means that the k_I's appearing are multiplied by p and the coefficient changes to ${}^{\mathrm{F}}\xi$ or ${}^{V^{-1}}\xi$. Again we see that the inequality still holds if we change C_2 to pC_2.

This shows the claim. □

Lemma 6.11. *Let e be a basic Witt differential satisfying an inequality indicating nonoverconvergence for constants C_1 and C_2, then so does ${}^{\mathrm{F}}e$ for the same constants.*

Proof. This is essentially the same argument as before, with the difference that this time we deal with strict inequalities in the other direction, so there is no need to increase the second constant. □

Proposition 6.12. *The map $1 - \mathrm{F}$ over $X = \mathbb{G}_m^d$ is surjective for étale topology.*

Proof. Let $\omega \in W^{\dagger}\Omega_{X/k}$. We have seen that up to étale localisation, there is $\eta \in W\Omega_{X/k}$ such that $\omega = (1-\mathrm{F})\eta$. We have to show that η is in fact overconvergent.

Write $\eta = \sum_{k,\mathscr{P},J} e(\xi, k, \mathscr{P}, J)$ as a unique sum of elementary Witt differentials and assume that it is not overconvergent. Then for all $C_1 > 0$ and $C_2 \in \mathbb{R}$ there is an element e appearing in the sum that violates the inequalities for overconvergence, or in other words satisfies the strict inequalities for nonoverconvergence and so do the elements ${}^{\mathrm{F}^i}e$, $i \in \mathbb{N}_0$ for the same constants C_1 and C_2.

Since $\omega = (1-\mathrm{F}) \sum_{k,\mathscr{P},J} e(\xi, k, \mathscr{P}, J)$ is overconvergent there must be C_1, C_2 for which the corresponding elements that violate the overconvergence inequality in the original sum cancel out after applying $1-\mathrm{F}$. Let e be one of these elements.

The image of e is $e - {}^{\mathrm{F}}e$. Due to the nature of the basic Witt differentials and the way Frobenius acts on the different types as pointed out in Remark 6.7.1, it is clear that e (and similarly ${}^{\mathrm{F}}e$) either remains and appears as a basic Witt differential in the unique decomposition of ω or is cancelled out by some basic Witt differential ${}^{\mathrm{F}}e'$ where e' is another basic Witt differential of the decomposition of η (similarly ${}^{\mathrm{F}}e$ appears or is cancelled out by a basic Witt differential e'' which appears in the decomposition of η).

Since we assumed that e is cancelled out after applying $1-\mathrm{F}$, the same must hold true for ${}^{\mathrm{F}}e$ which is subject to the same inequality. Hence $e'' = {}^{\mathrm{F}}e$ has to appear in the unique sum of η. By induction the basic Witt differentials ${}^{\mathrm{F}^i}e, i \in \mathbb{N}_0$ all appear in the unique sum of η. But a sum containing all of these elements cannot be convergent in the sense of Section 5.2. This is a contradiction to the uniqueness of the sum and therefore, η must be overconvergent to begin with. $\square$

Corollary 6.13. *The same is true for* $X = \operatorname{Spec} A[Y, Y^{-1}]$*, where* $A = k[X_1, \ldots, X_d]$.

Proof. This is just a simplified version of the previous assertion. $\square$

We have now established the following exact sequence for $X = \mathbb{G}_m^d$ for étale topology

$$0 \to W^{\dagger}\Omega^i_{X,\log} \to W^{\dagger}\Omega^i_X \xrightarrow{\mathrm{F}-1} W^{\dagger}\Omega^i_X \to 0.$$

We want to extend this result to general smooth schemes over k.

6.4.3 The map $1-\mathrm{F}$ over a smooth k-scheme

First we note, that we can reduce the general case to the case of a localised polynomial algebra. By a result of Kedlaya [Ked05] any smooth variety has a cover by standard étale affines as defined in [Ray70]. What is more, this cover can be chosen in a way that any finite intersection is again standard étale affine (see [Dav09, Proposition 4.3.1]). Let $A = k[X_1, \dots, X_d]$ and $f \in A$. In the proof of Theorem 1.8 in [DLZ11] the authors argue that it suffices to consider finite étale monogenic algebras over rings of the form A_f. In [DLZ11, Proposition 1.9] they reduce this further by stating

Proposition 6.14. *Let B be a finite étale and monogenic C-algebra, where C is smooth over a perfect field of char $p > 0$. Let $B = C[X]/(f(X))$ for a monic polynomial $f(X)$ of degree $m = [B : C]$ such that $f'(X)$ is invertible in B. Let $[x]$ be the Teichmüller of the element $X \mod f(X)$ in $W(B)$. Then we have for each $d \geqslant 0$ a direct sum decomposition of $W^\dagger(C)$-modules*

$$W^\dagger\Omega^d_{B/k} = W^\dagger\Omega^d_{C/k} \oplus W^\dagger\Omega^d_{C/k}[x] \oplus \cdots \oplus W^\dagger\Omega^d_{C/k}[x]^{m-1}.$$

Finally they prove that the overconvergent de Rham–Witt complex over a smooth k-scheme X is a complex of étale (and Zariski) sheaves on X. Thus we see that for our purposes as we seek a local result it is enough to consider the (overconvergent) de Rham–Witt complex over a localised polynomial algebra of the form A_f.

Now we proceed to calculate kernel and cokernel of the map $1-\mathrm{F}$.

Lemma 6.15. *Let $X = \operatorname{Spec} A_f$. The map $\mathrm{F}-1$ on $W^\dagger\Omega_{X/k}$ is surjective for étale topology.*

Proof. Consider the k-algebra $A[Y, Y^{-1}]$. There is a canonical surjection

$$\begin{array}{rcl} A[Y,Y^{-1}] & \to & A_f \\ Y & \mapsto & f. \end{array}$$

This induces by functoriality a surjection of the associated de Rham–Witt complexes $W\Omega_{A[Y,Y^{-1}]/k} \to W\Omega_{A_f/k}$. For quotients of polynomial algebras, an element of the corresponding de Rham–Witt complex is said to be overconvergent if there exist a lift of this element to the polynomial algebra, which is overconvergent. Therefore we have in fact a surjection of overconvergent de Rham–Witt complexes

$$W^{\dagger}\Omega_{A[Y,Y^{-1}]/k} \to W^{\dagger}\Omega_{A_f/k}.$$

Moreover, there is a commutative diagram

$$\begin{tikzcd} W^{\dagger}\Omega_{A[Y,Y^{-1}]/k} \arrow[r] \arrow[d, "\mathrm{F}-1"'] & W^{\dagger}\Omega_{A_f/k} \arrow[d, "\mathrm{F}-1"] \\ W^{\dagger}\Omega_{A[Y,Y^{-1}]/k} \arrow[r] & W^{\dagger}\Omega_{A_f/k} \end{tikzcd}$$

By Corollary 6.13 the vertical map on the left is surjective and thus, the same holds true for the one on the right. □

Lemma 6.16. *The kernel of* $\mathrm{F}-1$ *on* $W^{\dagger}\Omega_{X/k}$ *for* $\mathrm{X} = \operatorname{Spec} A_f$ *is* $W^{\dagger}\Omega_{X/k,\log}$.

Proof. By definition of the complex $W^{\dagger}\Omega_{X/k,\log}$ and the discussion above, it is contained in the kernel of $\mathrm{F}-1$. On the other hand, $W^{\dagger}\Omega_{X/k,\log} = W\Omega_{X/k,\log}$ and this is the kernel of $\mathrm{F}-1$ on $W\Omega_{X/k}$ without the overconvergence condition. Thence the restriction of $\mathrm{F}-1$ to the overconvergent subcomplex $W^{\dagger}\Omega_{X/k}$ must have the same kernel. □

Remark 6.16.1. This argument could have been applied to any smooth variety X over k.

Combining the arguments of the last two sections yields

Corollary 6.17. *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Then for all $i \in \mathbb{N}$ there is locally for étale topology a short exact sequence*

$$0 \to W^{\dagger}\Omega^{i}_{X/f,\log} \to W^{\dagger}\Omega^{i}_{X/k} \xrightarrow{\mathrm{F}-1} W^{\dagger}\Omega^{i}_{X/k} \to 0.$$

6.4.4 Chern classes into the Frobenius fixed part

Unfortunately the Frobenius morphism, and therefore also the morphism $1 - \mathrm{F}$, is only a ring homomorphism and not a morphism of complexes. The reason for this is that F does not commute with the differential; in fact the formula

$$d\,\mathrm{F} = p\,\mathrm{F}\,d \tag{6.4}$$

holds. Thus some modifications are required which are inspired by [Ill79, Corollaire I.3.29 and Théorème II.5.5].

Definition 6.18. *For each $m \geqslant 1$ we define an endomorphism of complexes*

$$\mathrm{F}_m : W\Omega^{\geqslant m} \to W\Omega^{\geqslant m},$$

where we use the naïve truncation, which is given in degree $i \geqslant m$ by $p^{i-m}\,\mathrm{F}$.

Looking at the commuting formula (6.4) of d and F, it is clear now that this definition gives indeed a morphism of complexes, and by extension the same holds true for $1 - \mathrm{F}_m$.

Illusie shows in [Ill79, Lemme I.3.30] that for all $r \geqslant 1$ and all $i \geqslant 0$, the morphism $1 - p^r\,\mathrm{F}$ is an automorphism of the proobject $W_\bullet\Omega^i_X$; hence (for example using the Mittag-Leffler condition) the induced map on $W\Omega^i_X$ is also an automorphism.

We consider now the restriction of theses morphisms to the overconvergent subobjects. We have already seen that multiplication by p and the Frobenius F map overconvergent elements to overconvergent elements. Thus for $m \geqslant 1$ there is an endomorphism of complexes

$$1 - \mathrm{F}_m : W^\dagger\Omega^{\geqslant m} \to W^\dagger\Omega^{\geqslant m},$$

and for $r \geqslant 1$ and $i \geqslant 0$ the morphism $1 - p^r\,\mathrm{F} : W^\dagger\Omega^i_X \to W^\dagger\Omega^i_X$ as restriction from the usual de Rham–Witt complex is injective.

Unfortunately, the argument from Proposition 6.12, where we show that $1 - \mathrm{F}$ is surjective, does not work in this case as subsequent multiplication of a generalised basic Witt differential by p^r for a fixed $r > 1$ creates an overconvergent sequence.

Together with the short exact sequence from Corollary 6.17 we obtain for a fixed $m \geqslant 1$ the following exact sequence of complexes:

$$0 \to W^{\dagger}\Omega^{m}_{X/k,\log}[-m] \to W^{\dagger}\Omega^{\geqslant m}_{X/k} \xrightarrow{1-\mathrm{F}_m} W^{\dagger}\Omega^{\geqslant m}_{X/k}. \tag{6.5}$$

However, this is enough for our purposes.

The exact sequence (6.5) induces an exact sequence on cohomology.

Proposition 6.19. *For X/k smooth and $m \in \mathbb{N}_0$, $i \in \mathbb{N}$ there is an exact sequence*

$$0 \to \mathrm{H}^{m}(X, W^{\dagger}\Omega^{i}_{\log}) \to \mathbb{H}^{m+i}(X, W^{\dagger}\Omega^{\geqslant i}) \xrightarrow{1-\mathrm{F}_i} \mathbb{H}^{m+i}(X, W^{\dagger}\Omega^{\geqslant i}).$$

Proof. We start with the exact sequence (6.5) and replace the rightmost object by the image of $1-\mathrm{F}_m$, which makes it into a short exact sequence. In the associated long exact sequence of cohomology, the connecting morphisms are obviously trivial, and after going back to the original complexes we obtain the above sequence for each i and m. □

In particular, we obtain the following result.

Corollary 6.20. *Let X/k be smooth and $m \in \mathbb{N}_0$, $i \in \mathbb{N}$. Then we have the identity*

$$\mathbb{H}^{m+i}(X, W^{\dagger}\Omega^{\geqslant i})^{1-\mathrm{F}_i} = \mathrm{H}^{m}(X, W^{\dagger}\Omega^{i}_{\log}).$$

The submodule $\mathbb{H}^{m+i}(X, W^{\dagger}\Omega^{\geqslant i})^{1-\mathrm{F}_i} \subset \mathbb{H}^{m+i}(X, W^{\dagger}\Omega^{\geqslant i})$ can be thought of as the Frobenius eigen module of eigenvalue $\frac{1}{p^m}$.

Recall that by construction the overconvergent Chern classes factor through the logarithmic differentials. Therefore the stated identity entails the subsequent corollary.

Corollary 6.21. *Let X/k be smooth. Then the overconvergent Chern classes constructed earlier can be written as*

$$c^{sc}_{ij} : K_j(X) \to \mathbb{H}^{2i-j}(X, W^{\dagger}\Omega^{\geqslant i})^{1-\mathrm{F}_i}.$$

Taking into account the γ-filtration, especially Proposition 6.6 (2), yields Chern classes on the γ-graded pieces of the algebraic K-groups.

Corollary 6.22. *Let X/k be smooth. There are overconvergent Chern classes*

$$c^{sc}_{ij} : \mathrm{gr}^{i}_{\gamma} K_j(X) \to \mathbb{H}^{2i-j}(X, W^{\dagger}\Omega^{\geqslant i})^{1-\mathrm{F}_i}.$$

CHAPTER 7

COMPARISON OF CHERN CLASSES

The purpose of this section is to show that in the case of a smooth and quasi-projective variety the overconvergent Chern classes from the previous section are compatible with the rigid Chern classes defined by Petrequin in [Pet03].

7.1 Rigid Chern classes

Let X be a proper variety over k and $\mathscr{V}$ a discrete valuation ring with residue field k. We choose a closed immersion $X \hookrightarrow \mathscr{Y}$, where $\mathscr{Y}$ is a formal scheme over $\mathrm{Spf}(\mathscr{V})$ smooth in a neighbourhood of X. Let $D = (\mathscr{L}, Z, s)$ be a good pseudo-divisor in the sense of Fulton:

Definition 7.1. *Let X be a variety. A pseudo-divisor on X is a triple $(\mathscr{L}, Z, s)$, where $\mathscr{L}$ is an invertible sheaf on X, $Z \subset X$ closed and s a trivialisation of the restriction $\mathscr{L}\,\big|_{X-Z}$. The closed subset Z is called the support of the pseudo-divisor. If the support is locally a zero-set of a section of $\mathscr{O}_X$, the pseudo-divisor $(\mathscr{L}, Z, s)$ is called good.*

Petrequin calculates the cohomology class associated to a pseudo-divisor using Čech cohomology. He shows that there is an affine cover $\mathfrak{U} = (\mathscr{U}_i)$ of $\mathscr{Y}$ such that the induced cover $\mathfrak{U}_X$ on X trivialises $\mathscr{L}$. If this cover is given by $\mathscr{U}_i = \mathrm{Spf}(\mathscr{A}_i)$, the induced cover on X is given by $X_i = \mathrm{Spec}(A_i)$, where $A_i = \mathscr{A}_i\,/I_i$. Since D is a good pseudo-divisor, one can moreover assume that $Z_i = Z \cap X_i$ is given by an equation $h_i \in A_i$. Let $U = X - Z$ and $U_i = X_i - Z_i$, and $j : U \hookrightarrow X$ the inclusion.

If $\varphi : \mathscr{O}_{X_i} \xrightarrow{\sim} \mathscr{L}\,\big|X_i$ is a trivialisation of $\mathscr{L}$, one sets

$$\phi_i = \varphi_i(1) \in \mathscr{L}\,\big|_{X_i}.$$

To this trivialisation we associate a cocycle $(u) \in Z^1(\mathfrak{U}_X\,\mathscr{O}_X^*)$ consisting of the transition maps

$$\phi_j = u_{ij}\phi_i,$$

where $u_{ij} \in (A_{ij})_{h_i h_j}$. To these data Petrequin associates a class in the group $\mathrm{H}^2\left(\mathfrak{U}_K, (\Omega^*_{]X[} \to j^\dagger_U \Omega^*_{]X[})_s\right)$ by constructing an element in $C^2\left(\mathfrak{U}_K, (\Omega^*_{]X[} \to j^\dagger_U \Omega^*_{]X[})_s\right)$, which has the decomposition

$$C^2(\mathfrak{U}_K, \mathscr{O}_{]X[}) \oplus C^1(\mathfrak{U}_K, \Omega^1_{]X[} \oplus j^\dagger_U \mathscr{O}_{]X[}) \oplus C^0(\mathfrak{U}_K, \Omega^2_{]X[} \oplus j^\dagger_U \Omega^1_{]X[}),$$

in terms of liftings $\tilde{u}_{ij}$ of u_{ij} to $\mathscr{A}_{ij}$. Seen as a rigid analytic function on $]X_{ij}[$, $\tilde{u}_{ij}$ restricts to an invertible element on X. In particular the middle (and in our case the only relevant) part $(\mu) \in C^1(\mathfrak{U}_K, \Omega^1_{]X[})$ of the expression sought is given by

$$\mu_{ij} = \frac{d\tilde{u}_{ij}}{\tilde{u}_{ij}}.$$

A calculation shows that this defines indeed a cycle, whose class in the Čech cohomology group $\check{\mathrm{H}}^2\left(]X[, (\Omega_{]X[} \to j^\dagger_U \Omega_{]X[})_s\right)$ denoted by $c_1(\mathscr{L}, Z, s)$ is independent of both the choice of the trivialisation and the choice of the lifting of (u) (see [Pet03, Proposition 3.10]). Its image in $\mathrm{H}^2_{Z,\mathrm{rig}}(X/K)$ is independent of the inclusion. This defines a group morphism

$$c_1 : \mathrm{Div}_Z(X) \to \mathrm{H}^2_{Z,\mathrm{rig}}(X/K),$$

which is functorial in X and can be extended by scalars.

Functoriality allows this to be generalised for open varieties.

Consider a (smooth) k-variety X and a vector bundle $\mathscr{E}$ of rank r over X. We denote by $\pi : \mathbb{P} = \mathbb{P}_{\mathscr{E}} \to X$ the associated projective bundle. Let

$$\xi = c_1^{\mathrm{rig}}(\mathscr{O}_{\mathbb{P}}(1)) \in \mathrm{H}^2_{\mathrm{rig}}(\mathbb{P}/K)$$

be the class of the good pseudo-divisor $(\mathscr{O}_{\mathbb{P}}(1), X, -)$ defined above. As we have pointed out this can be calculated by the Čech cocycle

$$\left(\frac{du}{u}\right) \in Z^2(\mathfrak{U}_K, \Omega^*_{]\mathbb{P}[}),$$

where $\mathfrak{U}$ is a covering trivialising $\mathscr{O}_{\mathbb{P}}(1)$ over $\mathscr{V}$. By [Pet03, Corollary 4.4] there is a projective bundle formula

$$\mathrm{H}^n_{\mathrm{rig}}(\mathbb{P}/K) \cong \bigoplus_{i=0}^{r-1} \mathrm{H}^{n-2i}_{\mathrm{rig}}(X/K)\xi^i.$$

In much the same spirit as Gillet's arguments, we then define Chern classes as the coefficients of the decomposition of ξ^r under this isomorphism

$$\xi^r = \sum_{i=1}^{r} (-1)^{i+1} c_i^{\mathrm{rig}}(\mathscr{E}) \xi^{r-i},$$

with $c_i^{\mathrm{rig}}(\mathscr{E}) \in \mathrm{H}^{2i}_{\mathrm{rig}}(X/K)$. This is well defined if we require $c_0^{\mathrm{rig}}(\mathscr{E}) = 1 \in \mathrm{H}^0_{\mathrm{rig}}(X/K)$.

As in the classical case this induces a theory of Chern classes for higher algebraic K-theory with coefficients in rigid cohomology

$$c_{ij}^{\mathrm{rig}} : K_j(X) \to \mathrm{H}^{2i-j}_{\mathrm{rig}}(X/K).$$

Proposition 7.2. *Let X/k be a smooth variety. The rigid Chern classes defined by Petrequin factor through Milnor K-theory via a morphism*

$$\mathrm{H}^i(X, \mathscr{K}^M_m) \to \mathrm{H}^{i+m}_{rig}(X/K). \tag{7.1}$$

Proof. We start with the case of infinite residue fields. If X is not proper let $j : X \hookrightarrow \overline{X}$ be a suitable compactification, which exists by [Pet03, Lemme 3.19], otherwise take $X = \overline{X}$. Furthermore, let $\overline{X} \hookrightarrow \mathscr{Y}$ be a closed immersion into a formal scheme over $\mathrm{Spf}(\mathscr{V})$ as before, and $]X[$ and $]\overline{X}[$ the tubes of X and $\overline{X}$ respectively in the generic fibre of $\mathscr{Y}$.

For a local section x of $\mathscr{O}_X^*$ choose a lift $\tilde{x}$ over $\mathscr{Y}$. This can be seen as a rigid analytic function on $]X[$ and $]\overline{X}[$ that restricts to an invertible element on X, which is therefore itself invertible (as rigid analytic function). We thus set

$$\mu = \frac{d\tilde{x}}{\tilde{x}}$$

and thereby define a local section of $\Omega^1_{]\overline{X}[}$ whose cocycle class is independent of the choice of lift. In the same manner, it is possible to assign to a local section $x_1 \otimes \cdots \otimes x_i \in \mathscr{O}_X^* \otimes \cdots \otimes \mathscr{O}_X^*$ a section

$$\mu_1 \cdots \mu_i = \frac{d\tilde{x}_1}{\tilde{x}_1} \cdots \frac{d\tilde{x}_i}{\tilde{x}_i}$$

of $\Omega^i_{]\overline{X}[}$. It is clear that

$$\frac{d\tilde{x}}{\tilde{x}} \frac{d(\tilde{1} - \tilde{x})}{(\tilde{1} - \tilde{x})} = 0$$

and therefore, as $\tilde{1} - \tilde{x}$ is a lift of $1 - x$, the classes of the symbols $\mu_1 \cdots \mu_i$ satisfy the Steinberg relation. Consequently, there are induced morphisms of cohomology groups

$$\mathrm{H}^m(X, \overline{\mathscr{K}}_i^M) \to \mathbb{H}^{i+m}\left( ]\overline{X}[, j^\dagger \Omega_{]\overline{X}[} \right) = \mathrm{H}_{\mathrm{rig}}^{i+m}(X/K), \tag{7.2}$$

and it is clear that it respects the multiplicative structure of the cohomology rings.

To see that this factors the rigid Chern classes, consider a vector bundle $\pi : \mathscr{E} \to X$ of constant rank n, and let $\mathbb{P} = \mathbb{P}(\mathscr{E})$ be the associated projective bundle. As we mentioned, there is a cover $\mathfrak{U} = (\mathscr{U}_i)$ of $\mathscr{Y}$ such that the induced cover $\mathfrak{U}_X = (U_i)$ of X trivialises the line bundle $\mathscr{O}_{\mathbb{P}}(1)$. To this trivialisation we can associate in the classical manner a Čech cocycle

$$(u) \in \mathrm{Z}^1(\mathfrak{U}_X, \mathscr{O}_X^*)$$

as u_{ij} on $U_i \cap U_j$, which calculates the first Chern class of $\mathscr{O}_{\mathbb{P}}(1)$ in $\mathrm{H}^1(\mathbb{P}, \overline{\mathscr{K}}_1^M)$. On the other hand, Petrequin shows that the first rigid Chern class of $\mathscr{O}_{\mathbb{P}}(1)$ in $\mathrm{H}^2_{\mathrm{rig}}(\mathbb{P}/K)$ is given by the Čech cocycle

$$\left(\frac{d\tilde{u}}{\tilde{u}}\right) \in Z^2(\mathfrak{U}_K, \Omega_{]\mathbb{P}[}),$$

and its class agrees with the image of the class of (u) under the morphism (7.2). In both cases the Chern classes $c_i^{\mathrm{rig}}(\mathscr{E}) \in \mathrm{H}^{2i}_{\mathrm{rig}}(X/K)$ and $c_i^M(\mathscr{E}) \in \mathrm{H}^i(X, \overline{\mathscr{K}}_i^M)$ are uniquely defined via a projective bundle formula using the same relations namely

$$\begin{aligned} c_0(\mathscr{E}) &= 0, \\ c_{i>r}(\mathscr{E}) &= 0, \\ \sum_{i=0}^{r} c_i(\mathscr{E}) c_1\left(\mathscr{O}_{\mathbb{P}}(1)\right)^{r-i} &= 0. \end{aligned}$$

Indeed, we have a commutative diagram of the form

$$\begin{tikzcd} \mathrm{H}^j(\mathbb{P}, \mathrm{K}_j^M) \arrow[r] & \mathrm{H}^j_{\mathrm{rig}}(\mathbb{P}/K) \\ \bigoplus_{i=0}^{n-1} \mathrm{H}^{j-i}(X, \overline{\mathscr{K}}_{j-i}^M) \cdot c_1^M(\mathscr{O}_{\mathbb{P}}(1))^i \arrow[u, "\sim"] \arrow[r] & \bigoplus_{i=0}^{n-1} \mathrm{H}^{i-2j}_{\mathrm{rig}}(X/K) \cdot c_1^{\mathrm{rig}}(\mathscr{O}_{\mathbb{P}}(1))^i \arrow[u, "\sim"'] \end{tikzcd}$$

The fact that the morphism (7.1) is compatible with multiplication shows that the rigid Chern classes c_i^{rig} factor through Milnor K-theory sheaves.

The same then holds for the higher Chern classes, and the diagram

$$\begin{tikzcd} K_j(X) \arrow[rr, "c_{ij}^{\mathrm{rig}}"] \arrow[dr, "c_{ij}^M"'] & & \mathrm{H}_{\mathrm{rig}}^{2i-j}(X/K) \\ & \mathrm{H}^{i-j}(X, \overline{\mathscr{K}}_i^M) \arrow[ur] & \end{tikzcd}$$

is commutative.

Now we come to the case of finite residue fields. It is easy to see that the algebraic de Rham complex Ω is continuous abelian sheaf, which disposes of a transfer on the big étale (as well as Zariski) site of schemes, in other words it is a continuous object of the category $\mathfrak{S}\mathfrak{T}_{\text{ét}}$. Thus by Corollary 2.14, the morphism above induces a morphism of cohomology groups for the improved Milnor K-sheaf

$$\mathrm{H}^m(X, \widehat{\mathscr{K}}_i^M) \to \mathbb{H}^{i+m}\left(]\overline{X}[, j^\dagger\Omega_{]\overline{X}[}\right) = \mathrm{H}_{\mathrm{rig}}^{i+m}(X/K). \tag{7.3}$$

Recall however that Rost's results state among other things that the cohomology of a cycle module can be calculated by using the associated complex, which means in particular that we have to evaluate them solely on fields—and on fields Kerz's usual and improved Milnor K-theories coincide.

We have seen that the morphism (7.1) factors the rigid Chern classes of Petrequin in the case of a scheme with infinite residue fields. As a consequence of the remark in the previous paragraph and of the uniqueness property in Corollary 2.14 used for the construction of the Milnor Chern classes in the case of finite residue fields and also for the construction of morphism (7.3), we conclude that the morphism (7.3) factors the rigid Chern classes in the case of finite residue fields, i.e., the diagram

$$\begin{tikzcd} K_j(X) \arrow[rr, "c_{ij}^{\mathrm{rig}}"] \arrow[dr, "c_{ij}^M"'] & & \mathrm{H}_{\mathrm{rig}}^{2i-j}(X/K) \\ & \mathrm{H}^{i-j}(X, \widehat{\mathscr{K}}_i^M) \arrow[ur] & \end{tikzcd}$$

is commutative. □

7.2 The comparison theorem between rigid and overconvergent cohomology

From now on, let X/k be smooth and quasi-projective and K the fraction field of $W(k)$. We will recall briefly the canonical comparison isomorphism

$$\mathrm{H}^i_{\mathrm{rig}}(X/K) \xrightarrow{\sim} \mathbb{H}^i\left(W^\dagger\Omega_{X/k}\right) \otimes \mathbb{Q}$$

between rigid and overconvergent de Rham cohomology constructed by Davis, Langer and Zink [DLZ11].

Let $X = \operatorname{Spec} A/k$ be smooth. Assume that there is an embedding of X in a formal scheme, more precisely, let $F = \operatorname{Spec} B$ be a smooth affine scheme over $W(k)$ such that $X \subset F$ is a closed subscheme, and $B \to A$ an epimorphism. Assume moreover that there is a homomorphism $\varkappa : B \to W(A)$ lifting $B \to A$. The triple $(X, F, \varkappa)$ is called a Witt frame; it is said to be overconvergent if the image of $\varkappa$ lies in $W^\dagger(A)$. Denote by $\hat{F}$ the completion of F in the ideal (p), and let $]X[_{\hat{F}}$ be the tubular neighbourhood of X in $\hat{F}_K$ as defined by Berthelot. There is a natural map

$$\Gamma(]X[_{\hat{F}}, \Omega_{]X[_{\hat{F}}}) \to W\Omega_{X/k} \otimes \mathbb{Q}$$

given in degree 0 as follows.

If $I = (f_1, \ldots, f_m)$ is the kernel of $B \to A$ and $\mathscr{A}$ the completion of B in I, let

$$\mathscr{A}_n = \mathscr{A}[T_1, \ldots, T_m]/(f_1^n - pT_1, \ldots, f_m^n - pT_m)$$

and $\hat{\mathscr{A}_n}$ its p-adic completion. Then $\hat{\mathscr{A}_n} \otimes \mathbb{Q}$ is an affinoid algebra and

$$\Gamma\left(]X[_{\hat{F}}, \mathscr{O}_{]X[_{\hat{F}}}\right) = \varprojlim \hat{\mathscr{A}_n} \otimes \mathbb{Q}\,.$$

Thus it is enough to define a compatible system of maps

$$\mathscr{A}_n \to W(A).$$

Note that $\varkappa$ maps I to ${}^VW(A)$ and that $W(A)$ is complete in the ideal ${}^VW(A)$. In particular, this means that $\varkappa(f_i) \in^V W(A)$, and consequently for $n \geqslant 2$

$$\varkappa(f_i^n) \in p^{n-1}({}^VW(A)),$$

and we can map $T_i \mapsto \frac{1}{p}\varkappa(f_i^n) \in W(A)$. This is well-defined because p is not a zero divisor in $W(A)$ and one obtains the compatible system of maps desired. One shows easily that this construction is functorial on the set of Witt frames.

By the universal property of Kähler differentials this is enough to define a map of de Rham complexes as described above. This is in fact an important fact that we will use later in Lemma 7.4.

Let $(X, F, \varkappa)$ be as before. Choose an embedding

$$F \subset \mathbb{A}^n_{W(k)} \subset \mathbb{P}^n_{W(k)}$$

and let Y be the closure of X in $\mathbb{P}^n_{W(k)}$. The authors explain in [DLZ11] how to construct a system of strict neighbourhoods of $]X[_{\hat{F}}$ in $]Y[_{\hat{Q}}$ where Q is the closure of F in $\mathbb{P}^n_{W(k)}$. This is vital because

$$X \to Y \to \hat{Q}$$

is a frame in the sense of rigid geometry, and the rigid cohomology of X can be calculated by

$$R\Gamma_{\text{rig}}(X) = R\Gamma(V, j^\dagger \Omega_V,)$$

where $V \subset F_K^{\text{an}}$ is a strict neighbourhood of $]X[_{\hat{F}}$ and $j : X \to \overline{X}$ is an open immersion which exists according to Nagata. It is a result of Berthelot that this definition is independent of the choice of V.

Assume now that $\varkappa$ is overconvergent. The next step is to define a map

$$\Gamma(V, j^\dagger \Omega_V) \to W^\dagger \Omega_{X/k} \otimes \mathbb{Q}$$

compatible with the map constructed above so that the diagram

$$\begin{tikzcd}
\Gamma(V, j^\dagger \Omega_V) \arrow[r] \arrow[d] & W^\dagger \Omega_{X/k} \otimes \mathbb{Q} \arrow[d] \\
\Gamma\left( ]X[_{\hat{F}}, \Omega_{]X[_{\hat{F}}} \right) \arrow[r] & W\Omega_{X/k} \otimes \mathbb{Q}
\end{tikzcd}$$

commutes.

The first case to consider is where F is the affine space over $W(k)$. For $\lambda < 1$ and $\eta = p^{-\frac{1}{r}}$ consider the strict neighbourhood $U_{\lambda\eta}$ of $]X[_{\hat{F}}$ given by the affinoid algebra

$$\mathcal{T}_{\lambda\eta} = K\left\langle \lambda X_1, \dots, \lambda X_n, T_1, \dots, T_m \right\rangle / (f_1^r - pT_1, \dots, f_m^r - pT_m).$$

Above we have seen that there is a homomorphism $\mathcal{T}_{\lambda\eta} \to W(A) \otimes \mathbb{Q}$. If we denote by ξ_i the image of X_i under this morphism, then $f_j(\xi_1, \dots, \xi_n) \in^V W(A)$; therefore there is ρ_j such that

$$f_j(\xi_1, \dots, \xi_n) =^V \rho_j.$$

The relations show that for $r \geqslant 3$ the image of T_j is

$$\frac{({}^V\rho_j)^r}{p} = p^{r-2}({}^V(\rho_j^r)).$$

This defines the morphisms, as an element $\mathfrak{p} = \sum a_{I,J} \underline{X}^I \underline{T}^J$ is mapped to

$$\sum a_{IJ} p^{(r-2)|J|} \underline{\xi}^I \left({}^V(\underline{\rho}^r)\right)^J.$$

Davis, Langer and Zink show that this is indeed overconvergent, i.e., it is an element of $W^\dagger(A) \otimes \mathbb{Q}$.

This can be extended to a general overconvergent Witt frame $(X, F, \varkappa)$. Showing that the so defined morphism factors naturally through $R\Gamma(V, j^\dagger\Omega_V)$ completes the construction of a morphism

$$R\Gamma_{\mathrm{rig}}(X) \to W^\dagger\Omega_{X/k} \otimes \mathbb{Q}$$

for an overconvergent Witt frame $(X, F, \varkappa)$. What is more, it can be shown that this is a quasi-isomorphism.

In order to globalise this, it is necessary to use dagger spaces. This means that locally $\mathcal{T}_{\lambda\eta} = K\langle \lambda X_1, \dots, \lambda X_n, T_1, \dots, T_m\rangle / (f_1^r - pT_1, \dots, f_m^r - pT_m)$ is replaced over a suitable extension $\tilde{K}/K$ by

$$\tilde{\mathcal{T}}_{\lambda\eta} = \tilde{K}\langle \lambda X_1, \dots, \lambda X_n, T_1, \dots, T_m\rangle / (f_1 - p^{\frac{1}{r}} T_1, \dots, f_m - p^{\frac{1}{r}} T_m).$$

One can rewrite the above constructed morphism in terms of dagger spaces

$$\Gamma\left(]X[^\dagger_{\hat{F}}, \Omega_{]X[^\dagger_{\hat{F}}}\right) \to W^\dagger\Omega_{X/k} \otimes \mathbb{Q}$$

and since $R\Gamma\left(]X[^\dagger_{\hat{F}}, \Omega_{]X[^\dagger_{\hat{F}}}\right) = R\Gamma_{rig}(X)$, this induces the same morphism

$$R\Gamma_{\mathrm{rig}}(X) \to W^\dagger\Omega_{X/k} \otimes \mathbb{Q}\,.$$

Let X be a smooth and quasi-projective variety over k. Thus X has a covering by standard smooth neighbourhoods. With an open embedding $X \to \operatorname{Proj} \mathcal{S}$ consider a finite cover $X = \bigcup X_i$, where $X_i = D^+(h_i) = \operatorname{Spec} A_i$ with $h_i \in \mathcal{S}$. For a multi-index $J = \{i_1, \dots, i_t\}$ set

$$X_J = X_{i_1} \cap \ldots \cap X_{i_t} = \operatorname{Spec} A_J,$$

where A_J is a suitable localisation of A_{i_1}. This is again standard smooth.

Let A be a standard smooth algebra over k represented as $k[X_1, \dots, X_n]/(f_1, \dots, f_m)$. Choose arbitrary liftings $\tilde{f}_1, \dots, \tilde{f}_m \in W(k)[X_1, \dots, X_n]$, and let B be a localisation of $W(k)[X_1, \dots, X_n]/(\tilde{f}_1, \dots, \tilde{f}_m)$ with respect to $\det\left(\frac{\partial \tilde{f}_i}{\partial X_j}\right)$, then B is standard smooth over $W(k)$ lifting A and therefore giving a special frame $(\operatorname{Spec} A, \operatorname{Spec} B)$. In this way we obtain special frames (X_i, F_i) for the cover of X. Using the simplicial structure associated to this covering the authors show in [DLZ11]

Theorem 7.3. *Let X be a smooth quasi-projective scheme over k. Then there is a natural quasi-isomorphism*

$$R\Gamma_{rig}(X) \xrightarrow{\sim} R\Gamma\left(X, W^\dagger\Omega_{X/k}\right) \otimes \mathbb{Q}\,. \tag{7.4}$$

Lemma 7.4. *The morphism (7.4) of Davis, Langer and Zink is compatible with the multiplicative structure on both sides: there is a commutative diagram*

$$\begin{tikzcd}
R\Gamma_{rig}(X) \otimes^L R\Gamma_{rig}(X) \arrow[r] \arrow[d] & \left(R\Gamma(X, W^\dagger\Omega_{X/k}) \otimes \mathbb{Q}\right) \otimes^L \left(R\Gamma(X, W^\dagger\Omega_{X/k}) \otimes \mathbb{Q}\right) \arrow[d] \\
R\Gamma_{rig}(X) \arrow[r] & \left(R\Gamma(X, W^\dagger\Omega_{X/k}) \otimes \mathbb{Q}\right)
\end{tikzcd}$$

where the tensor is taken over $W(k)$, the vertical maps represent the product and the upper horizontal map is given by the tensor product of the morphism (7.4).

Proof. Recall that $R\Gamma_{\mathrm{rig}}(X) = R\Gamma(V, j^\dagger\Omega_V)$, where V is a strict neighbourhood as above. Thus the left vertical map is given via the structure of Ω_V as differentially graded algebra. The same goes for the vertical map on the right hand side. Moreover, the comparison morphism (7.4) is by construction a morphism of differentially graded algebras, and consequently the diagram commutes. $\square$

The construction will become more explicit in the appendix, where we calculate it for the projective space.

7.3 Comparison with rigid Chern classes

Proposition 7.5. *Let X be as above. The morphism (7.1) of cohomology groups $\mathrm{H}^i(X, \mathscr{K}^M_m) \to \mathrm{H}^{i+m}_{rig}(X/K)$ factors through $\mathbb{H}^{i+m}(X, W^\dagger\Omega)$.*

Proof. Assume at first that X has infinite residue fields. As X is quasi-projective, we can choose an open embedding $X \to \operatorname{Proj} \mathcal{S}$, where $\mathcal{S}$ is a finitely generated graded algebra over k. Davis, Langer and Zink point out that this enables us to consider finite coverings $\mathfrak{U}_X = \{U_i\}$ of X fine enough such that the $U_i = \operatorname{Spec} A_i$ are standard smooth affines as well as their intersections. What is more, if we are given a global section $\overline{u} \in \mathscr{O}^*_X$, we can choose the covering in a way that it represents a trivialisation for $\overline{u}$ given in local coordinates by

$$\overline{u}_{ij} \qquad \text{on } U_i \cap U_j.$$

Further refinements allow us to consider m such sections $\overline{u}^{(1)}, \ldots, \overline{u}^{(m)}$ of $\mathscr{O}^*_X$, each of which are given in local coordinates $\overline{u}^{(k)}_{ij}$.

Thus we can consider a section of $\overline{\mathscr{K}}^M_m$ locally given by

$$\{\overline{u}^{(1)}_{ij}, \ldots, \overline{u}^{(m)}_{ij}\} \qquad \text{on } U_i \cap U_j.$$

Its image in $W^\dagger\Omega\,[1]$ under $d\log$ is

$$\frac{d[\overline{u}^{(1)}_{ij}]}{[\overline{u}^{(1)}_{ij}]} \cdots \frac{d[\overline{u}^{(m)}_{ij}]}{[\overline{u}^{(m)}_{ij}]} \qquad \text{on } U_i \cap U_j.$$

As in [DLZ11] we can choose for each A_i a standard smooth lift B_i over $W(k)$ with a fixed Frobenius lift such that for the lift $u^{(l)}_{ij}$ of each $\overline{u}^{(l)}_{ij}$

$$F(u^{(l)}_{ij}) = (u^{(l)}_{ij})^p,$$

and a homomorphism $\varkappa_i : B_i \to W(A_i)$, induced by F, which lifts $B_i \to A_i$ such that the image is overconvergent, thereby giving an overconvergent frame $(U_i, F_i, \varkappa_i)$ where $F_i = \operatorname{Spec} B_i$. By choosing the covering fine enough, we ensure that all intersections are again standard smooth affine and give rise to overconvergent frames. We denote the intersections by $U_I = \bigcap_{i \in I} U_i$ and $U_I = A_I$, where I is a multi-index.

Let as before $j : X \hookrightarrow \overline{X}$ be a suitable compactification and let $\overline{X} \hookrightarrow \mathscr{Y}$ be a closed immersion into a formal scheme over $\mathrm{Spf}(W(k))$. Denote by $\mathscr{U}_i = \mathrm{Spf}\,\mathscr{A}_i$ the formal completion of F_i along U_i, which cover the image of X in $\mathscr{Y}$. This can be completed to a covering $\mathfrak{U}$ of $\mathscr{Y}$ that induces the covering $\mathfrak{U}_X$. Further denote by $\mathfrak{U}_K$ the induced cover of the rigid generic fibre. Analogue to above we use the notations $\mathscr{U}_I$, $\mathscr{A}_I$ etc.

Again we choose liftings $\tilde{u}_{ij}^{(1)}, \ldots, \tilde{u}_{ij}^{(m)} \in \mathscr{A}_{ij}$ of the local sections $\overline{u}_{ij}^{(1)}, \ldots, \overline{u}_{ij}^{(m)}$. Then

$$\frac{d\tilde{u}_{ij}^{(1)}}{\tilde{u}_{ij}^{(1)}} \cdots \frac{d\tilde{u}_{ij}^{(m)}}{\tilde{u}_{ij}^{(m)}}$$

is a local section of $\Omega_{]\overline{X}[}$ and the image of $\{\overline{u}_{ij}^{(1)}, \ldots, \overline{u}_{ij}^{(m)}\} \in \overline{\mathscr{K}}_m^M$ under the map defined in Proposition 7.2. The way these local sections were obtained implies that they glue to a global section $\frac{d\tilde{u}}{\tilde{u}}$ if we use dagger spaces.

The task now is to check that the image of $\frac{d\tilde{u}_{ij}^{(1)}}{\tilde{u}_{ij}^{(1)}} \cdots \frac{d\tilde{u}_{ij}^{(m)}}{\tilde{u}_{ij}^{(m)}}$ under the comparison morphism of Davis,Langer and Zink is compatible with $\frac{d[\overline{u}_{ij}^{(1)}]}{[\overline{u}_{ij}^{(1)}]} \cdots \frac{d[\overline{u}_{ij}^{(m)}]}{[\overline{u}_{ij}^{(m)}]}$.

Recalling that the $\overline{u}_{ij}^{(l)} \in A_{ij}$ and $\tilde{u}_{ij}^{(l)} \in B_{ij}$ are local coordinates and that we chose the Frobenius lift in a particular way, we see that the image of $\tilde{u}_{ij}^{(l)}$ under $\varkappa_{ij}$ is the Teichmüller lift $\left[\overline{u}_{ij}^{(l)}\right]$ (cf. [Dav09, Proposition 2.2.2]). By the construction in [DLZ11] it follows that the map

$$\Gamma\left(]U_i[_{\mathscr{U}_i}, \Omega_{]U_i[_{\mathscr{U}_i}}\right) \to W\Omega_{U_i} \otimes \mathbb{Q},$$

which is as a local map based upon the comparison map between the affine comparison morphism between Monsky–Washnitzer, and overconvergent cohomology sends the class of $\frac{d\tilde{u}_{ij}^{(l)}}{\tilde{u}_{ij}^{(l)}}$ to the class of $\frac{d[\overline{u}_{ij}^{(l)}]}{[\overline{u}_{ij}^{(l)}]}$ for all i, j, l. Although a priori this depends on the choice of Frobenius lift, Davis shows in [Dav09, Corollary 4.1.13] that the comparison map is in fact independent of it.

This morphism being the basis of the comparison morphism, we see that $\frac{d\tilde{u}_{ij}^{(l)}}{\tilde{u}_{ij}^{(l)}}$ is still mapped to $\frac{d[\overline{u}_{ij}^{(l)}]}{[\overline{u}_{ij}^{(l)}]}$ after passing to dagger spaces in order to glue. In particular, the Čech cocycle of rigid cohomology

$$\left(\frac{d\tilde{u}^{(l)}}{\tilde{u}^{(l)}}\right) \in Z^2(\mathfrak{U}_K, \Omega_{]X[}))$$

is sent to the Čech cocycle of overconvergent cohomology

$$\left(\frac{d[\overline{u}^{(l)}]}{[\overline{u}^{(l)}]}\right) \in Z^2(\mathfrak{U}, W^\dagger\Omega_X),$$

for varying l. Thus, the same holds true for $\frac{d\tilde{u}^{(1)}}{\tilde{u}^{(1)}} \cdots \frac{d\tilde{u}^{(m)}}{\tilde{u}^{(m)}}$, which accordingly is sent to $\frac{d[\overline{u}^{(1)}]}{[\overline{u}^{(1)}]} \cdots \frac{d[\overline{u}^{(m)}]}{[\overline{u}^{(m)}]}$.

This shows that the induced morphism $\mathrm{H}^i(X, \overline{\mathscr{K}}^M_m) \to \mathrm{H}^{i+m}_{\mathrm{rig}}(X/K)$ factors indeed through $\mathbb{H}^{i+m}(X, W^\dagger\Omega)$, and the diagram

$$\begin{tikzcd}
\mathrm{H}^i(X, \overline{\mathscr{K}}^M_m) \arrow[rr] \arrow[dr] & & \mathrm{H}^{i+m}_{\mathrm{rig}}(X/K) \arrow[r, "\sim"] & \mathbb{H}^{i+m}(X, W^\dagger\Omega) \otimes \mathbb{Q} \\
& \mathbb{H}^{i+m}(X, W^\dagger\Omega) \arrow[urr] & &
\end{tikzcd}$$

commutes.

A similar reasoning can be applied in the case of finite residue fields. The morphisms $\mathrm{H}^i(X, \widehat{\mathscr{K}}^M_m) \to \mathrm{H}^{i+m}_{\mathrm{rig}}(X/K)$ and $\mathrm{H}^i(X, \widehat{\mathscr{K}}^M_m) \to \mathbb{H}^{i+m}(X, W^\dagger\Omega)$ are both deduced from the corresponding morphisms for Kerz's usual Milnor K-theory, and they are both unique by the uniqueness property of Corollary 2.14. As in the infinite case $\mathrm{H}^i(X, \overline{\mathscr{K}}^M_m) \to \mathrm{H}^{i+m}_{\mathrm{rig}}(X/K)$ factors through $\mathbb{H}^{i+m}(X, W^\dagger\Omega)$ via the comparison isomorphism

$$\mathrm{H}^n_{\mathrm{rig}}(X/K) \to \mathbb{H}^n(X, W^\dagger\Omega) \otimes \mathbb{Q}$$

of Davis, Langer and Zink, the same holds true for the finite case, and we obtain a commutative diagrams above with $\overline{\mathscr{K}}^M_m$ replaced by $\widehat{\mathscr{K}}^M_m$. □

Now we can conclude with a comparison of rigid and overconvergent Chern classes.

Theorem 7.6. *Let X be a smooth quasi-projective scheme over k. The overconvergent Chern classes for X defined here are compatible with the rigid Chern classes defined in [Pet03].*

Proof. We use the fact that the rigid and the overconvergent Chern classes factor through the Milnor K-sheaf. Consider the following diagram

$$\begin{tikzcd}
 & & \mathrm{H}^{2j-i}_{\mathrm{rig}}(X/K) \\
K_j(X) \arrow[r, "c_{ij}^{M}"] \arrow[urr, "c_{ij}^{\mathrm{rig}}"] \arrow[drr, "c_{ij}^{\mathrm{sc}}"'] & \mathrm{H}^{i-j}(X, \mathscr{K}^M_i) \arrow[ur] \arrow[dr] & \\
 & & \mathbb{H}^{2i-j}(X, W^\dagger\Omega) \arrow[u]
\end{tikzcd}$$

where all the triangles commute: the upper left one by Proposition 7.2, the lower left one by construction and the right one by the previous lemma. Given that all morphisms involved are compatible with products, this shows that the Chern classes in question are indeed compatible.

□

CHAPTER 8

OVERCONVERGENT CYCLE CLASSES

Recall that one can interpret classical Chow groups in terms of Milnor K-sheaves

$$\mathrm{CH}^i(X) = \mathrm{H}^i(X, \mathscr{K}_i^M),$$

which is also known as Bloch's formula. In this sense, the morphism of cohomology groups (6.1) constitutes for $i = m$ a cycle class on the classical Chow groups

$$\eta_{\mathrm{sc}}^i : \mathrm{CH}^i(X) \to \mathbb{H}^{2i}(X, W^\dagger\Omega_X^{\geqslant i}).$$

Remark 8.0.1. As a consequence of our comparison in Proposition 7.6, one can conclude that these overconvergent cycle classes are compatible with Petrequin's rigid Chern classes in [Pet03, Section 6].

Moreover, we can restrict the Chern classes $c_{ij}^{\mathrm{sc}} : K_j(X) \to \mathrm{H}^{2i-j}(X, W^\dagger\Omega)$ to the graded pieces $gr_\gamma^i K_j(X)$ of algebraic K-theory induced by the γ-filtration (see the Section 6.3 on this subject)

$$\eta_{\mathrm{sc}}^{ij} : gr_\gamma^i K_j(X) \to \mathrm{H}^{2i-j}(X, W^\dagger\Omega),$$

and it is well known that the $gr_\gamma^i K_j(X)$ are rationally isomorphic to Bloch's higher Chow groups $\mathrm{CH}^i(X, j)$. The goal of this section is to extend this morphism to an integral morphism of higher cycle classes

$$\eta_{\mathrm{sc}}^{ij} : \mathrm{CH}^i(X, j) \to \mathrm{H}^{2i-j}(X, W^\dagger\Omega^{\geqslant i})$$

compatible with our overconvergent Chern classes.

8.1 Bloch's higher Chow groups

Let k be a field. Bloch originally defined his higher Chow groups for equidimensional schemes of finite type over k [Blo86b], but the definition can be made for general equidimensional schemes. In the case when X/k is smooth, separated and k perfect, the definition is equivalent to Voevodsky's motivic cohomology theory.

Denote by Δ_k^N the standard algebraic N-simplex

$$\Delta_k^N := \operatorname{Spec} k[t_0, \ldots, t_n]/(\sum t_i - 1)$$

and by Δ_X^* the cosimplicial scheme given by

$$N \mapsto X \times_k \Delta_k^N.$$

The faces of Δ_X^N are defined by equations of the form $t_{i_1} = \cdots = t_{i_r} = 0$. Let $z_r(X, i)$ be the subgroup of the cycles of dimension $r + i$ generated by the set of irreducible dimension $r + i$-subschemes of Δ_X^i that intersect all faces properly (i.e., that intersect each dimension r-face in dimension $\leqslant r + r$). Bloch's simplicial group is then given by

$$i \mapsto z_r(X, i).$$

The Chow groups (with respect to dimension) are then the homology groups of the associated complex

$$\mathrm{CH}_r(X, i) = \mathrm{H}_i(z_r(X, *)).$$

In the case when X is equidimensional it is more convenient to label the complexes by codimension and define

$$\mathrm{CH}^r(X, i) = \mathrm{H}_i(z^r(X, *)),$$

where $z^r(X, i) = z_{n-r}(X, i)$ if X is of dimension n. We may extend the definition of $z^r(X, i)$ to arbitrary smooth schemes by taking the direct sum over the irreducible components.

The relation to Voevodsky's motivic theory in the case of smooth, separated schemes over a perfect field is

$$\begin{aligned} \mathrm{H}^i(X, \mathbb{Z}(r)) &= \mathrm{CH}^r(X, 2r - i) \\ \mathrm{CH}^r(X, i) &= \mathrm{H}^{2r-i}(X, \mathbb{Z}(r)). \end{aligned}$$

The complexes $z_r(X, *)$ are covariant for proper morphisms and contravariant for flat equidimensional morphisms so that it is possible to sheafify them (for étale or Zariski topology) $\mathscr{Z}_r(X,*)$. Similarly for the codimension complexes we get $\mathscr{Z}^r(X,*)$.

Bloch established the following list of properties for the groups $\mathrm{CH}^r(X,i)$ [Blo86b, ¶2]:

1. **Functoriality**. As is apparent from the remark above, they are covariant (with shift of codimension index) and contravariant for flat maps, or for all maps if the target is smooth.
2. **Homotopy.** Let $p : \mathbb{A}^1 \times X \to X$ be the projection. Then
$$\mathrm{CH}^r(X,i) = \mathrm{CH}^r(\mathbb{A}^1 \times X, i).$$
3. **Localisation**. Let $Y \subset X$ be a closed subscheme of pure codimension c and $U = X - Y$. Then there is a long exact sequence
$$\cdots \to \mathrm{CH}^*(U,i+1) \to \mathrm{CH}^{*-c}(Y,i) \to \mathrm{CH}^*(X,i) \to \mathrm{CH}^*(U,i) \to \cdots$$
4. **Degree zero**. The higher Chow groups coincide in degree zero with the classical Chow groups
$$\mathrm{CH}^r(X,0) = \mathrm{CH}^r(X).$$
5. **Local to global spectral sequence**. One has the equality
$$\mathrm{CH}^r(X,i) = \mathbb{H}^{-i}(X, \mathscr{Z}^r(X,*)).$$
In particular, there is a spectral sequence
$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{CH}^r(-q)) \Rightarrow \mathrm{CH}^r(X,-p-q),$$
where again $\mathscr{CH}^r(i)$ is the Zariski sheaf associated to the presheaf
$$U \mapsto \mathrm{CH}^r(U,i).$$
6. **Multiplicativity**. For schemes X and Y there is an exterior product
$$\mathrm{CH}^r(X,i) \otimes \mathrm{CH}^s(Y,j) \to \mathrm{CH}^{r+s}(X \times Y, i+j).$$
For smooth X and $Y = X$ pulling back along the diagonal yields a product structure
$$\mathrm{CH}^r(X,i) \otimes \mathrm{CH}^s(X,j) \to \mathrm{CH}^{r+s}(X,i+j)$$
and makes $\mathrm{CH}^*(X,*)$ into a ring.

7. **Chern classes**. For a rank n vector bundle $E \to X$ there are well defined operators for $1 \leqslant i \leqslant n$

$$\underline{c}_i(E) : \mathrm{CH}^a(X,b) \to \mathrm{CH}^{a+i}(X,b)$$

that satisfy the usual functoriality properties, and one can define

$$c_i(E) = \underline{c}_i(E)(X) \in \mathrm{CH}^i(X,0).$$

8. **Relationship with algebraic K-theory**. There is a rational equivalence

$$\mathrm{CH}^r(X,i) \otimes \mathbb{Q} \cong gr^r_\gamma K_i(X) \otimes \mathbb{Q}\,.$$

9. **Codimension 1**. For regular X there are the equalities

$$\mathrm{CH}^1(X,i) = \begin{cases} \mathrm{Pic}(X) & \text{if} \quad i = 0 \\ \Gamma(X, \mathscr{O}^*_X) & \text{if} \quad i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

10. **Gersten conjecture**. For X/k smooth there are flasque resolutions

$$0 \to \mathscr{C}\mathscr{H}^r_X(q) \to \bigoplus_{x \in X^{(0)}} \mathrm{CH}^r(x,q) \to \bigoplus_{x \in X^{(1)}} \mathrm{CH}^{r-1}(x,q-1) \to \cdots \quad .$$

In particular

$$\mathrm{CH}^r(X) = \mathrm{H}^r(X, \mathscr{C}\mathscr{H}^r(r)).$$

11. **Finite coefficients and the étale topology**. Let $\mathscr{Z}^*_{\text{ét}}(*)$ be the complex of étale sheaves associated to the codimension complex, n prime to the characteristic of the ground field k and $\pi : X \to \operatorname{Spec} k$ the structure map. Then

$$\pi^*(\mathscr{Z}^*_{\operatorname{Spec} k,\text{ét}}(*) \otimes \mathbb{Z}\,/\,n\,\mathbb{Z}) \to \mathscr{Z}^*_{X,\text{ét}}(*) \otimes \mathbb{Z}\,/\,n\,\mathbb{Z}$$

is a quasi-isomorphism.

8.2 Higher cycle classes with integral coefficients in the Milnor K-sheaf

Let X be smooth over a perfect field k of characteristic $p > 0$.

In [Blo86b, ¶4] Bloch defines higher cycle classes from higher Chow groups into reasonable bigraded cohomology theories

$$\mathrm{CH}^b(X,n) \to \mathrm{H}^{2b-n}(X,b).$$

Namely, the cohomology theory has to be the hypercohomology of a complex which is contravariant. By replacing it with its Godement resolution, it can be assumed to be built of acyclic sheaves. Moreover, Bloch assumes that for this theory one can define cohomology groups with supports, such that it satisfies a localisation sequence and that it satisfies homotopy invariance, the existence of a cycle class for subschemes of pure codimension and weak purity.

In order to construct higher cycle classes into the overconvergent cohomology groups, we take again the detour over Milnor K-theory using Rost's axiomatic. In particular we will not have to worry about the size of the residue fields of X.

Using Bloch's method we define for a fixed integer b a cycle map

$$\eta_M^{bn} : \mathrm{CH}^b(X,n) \to \mathrm{H}^{b-n}(X, \mathscr{K}_b^M),$$

which in turn induces a cycle map η_{sc}^{bn} into the overconvergent integral cohomology groups. We start by recalling some facts of cycle modules that hold in particular for the one associated to the Milnor K-groups.

Lemma 8.1. *The cohomology groups of the Milnor K-sheaves satisfy the conditions required by Bloch in his construction of cycle classes.*

Proof. We check the properties one by one.

1. **Calculated by a complex**. As we have seen in Corollary 3.16 and the subsequent remarks that according to Rost [Ros96] the sheaf cohomology of $\mathscr{K}_b^M$ over X can be calculated by the cohomology of the associated cycle complex $C^*(X; K_*^M, b)$
$$A^p(X; K_*^M, b) = \mathrm{H}^p(X, \mathscr{K}_b^M).$$

2. **Localisation Sequence**. For a closed subscheme $i : Y \hookrightarrow X$ let $C^*_Y(X, K^M_*, b)$ be the "cycle complex with supports" defined by

$$C^p_Y(X; K^M_*, b) = \coprod_{\substack{x \in X^{(p)} \\ x \in Y}} K^M_{b-p}(x),$$

and $A^n_Y(X; K^M_*, b)$ the associated cohomology with supports. Recall that there is a long exact sequence (3.2) for a closed subscheme $i : Y \hookrightarrow X$ and the associated immersion $j : U = X \backslash Y \to X$

$$\xrightarrow{\partial} A_p(Y; K^M_*, b) \xrightarrow{i_*} A_p(X; K^M_*, b) \xrightarrow{j_*} A_p(U; K^M_*, b) \xrightarrow{\partial} \quad .$$

This is in fact a localisation sequence: by definition we have Poincaré duality style equalities (see also the Appendix Section B)

$$\begin{aligned} A_{n-p}(Y; K^M_*, n-b) &= A^p_Y(X; K^M_*, b) \\ A_{n-p}(X; K^M_*, n-b) &= A^p(X; K^M_*, b), \end{aligned}$$

where n is the relative dimension of X over k. Thus the long exact sequence of homology above induces along exact sequence of cohomology

$$\xrightarrow{\partial} A^p_Y(X, K^M_*, b) \xrightarrow{i^*} A^p(X; K^M_*, b) \xrightarrow{j^*} A^p(U; K^M_*, b) \xrightarrow{\partial} A^{p+1}_Y(X; K^M_*, b) \to$$

and by the pointwise definition of cycle complexes, this sequence satisfies the usual functorial properties.

3. **Homotopy invariance**. According to Rost (equation 3.4), the cohomology groups $A^p(X; K^M_*, b)$ satisfy homotopy invariance

$$A^p(X; K^M_*, b) \cong A^p(X \times \mathbb{A}^1; K^M_*, b).$$

4. **Cycle class**. This point holds not for cycle modules in general but rather for Milnor K-theory. Let $Y \subset X$ be of pure codimension b. Similarly to the discussion above there is an isomorphism

$$A^b_Y(X; K^M_*, b) \cong A^0(Y; K^M_*, 0),$$

and the right hand sight is isomorphic to the zero cohomology group of $\mathscr{K}_0^M$ on Y. But for any ring $K_0^M(A) = \mathbb{Z}$, and there is a well-defined class

$$[Y] \in A_Y^b(X; K_*^M, b)$$

that corresponds to the identity, which by construction is contravariant functorial with respect to the pull-back of cycles.

5. **Weak purity**. Let $Y \subset X$ be of pure codimension r. There is an isomorphism

$$A_Y^p(X; K_*^M, b) \cong A_{n-p}(Y; K_*^M, b-n)),$$

where again n is the dimension of X. The right-hand side is zero of $n - p$ is greater than the dimension of Y. Consequently the left-hand side is zero if $p < r$.

□

Now we can go step by step through the construction of cycle classes. Using that the cohomology of the Milnor K-sheaf can be calculated by a complex, we see that the usual diagram of simplices

$$X \rightrightarrows X \times \Delta^1 \rightrightarrows X \times \Delta^2 \ \vdots \ \cdots \tag{8.1}$$

yields a double complex

$$C(X \times \Delta^\bullet; K_*^M, \bullet).$$

The first sheet of the spectral sequence associated to this double complex is given by

$$E_1^{pq} = A^q(X \times \Delta^{-p}; K_*^M, b).$$

Note the appearance of a sign at the index p on the right sight. This is due to the fact that the functor A^p is contravariant in its first place and the introduction of a sign aligns the induced morphisms by the natural maps of (8.1) with the required structure of a spectral sequence. The sheet E_1^{pq} is therefore by construction only nonzero for $p \leqslant 0$. In order to calculate the second sheet, we fix q and look at the associated bounded complex

$$\begin{aligned} \cdots \quad & A^q(X \times \Delta^{-p}; K_*^M, b) \xrightarrow{d_1^{pq}} A^q(X \times \Delta^{-(p+1)}; K_*^M, b) \to \cdots \\ & \to A^q(X \times \Delta^1; K_*^M, b) \xrightarrow{d_1^{1q}} A^q(X; K_*^M, b) \to 0, \end{aligned}$$

where the boundary morphisms

$$d_1^{pq} : E_1^{pq} = A^q(X \times \Delta^{-p}; K_*^M, b) \to E_1^{p+1,q} = A^q(X \times \Delta^{-p-1}; K_*^M, b)$$

are induced by the pull-backs of the maps in (8.1). By the homotopy invariance of the cohomology A^q,

$$A^q(X \times \Delta^{-p}; K_*^M, b) \cong A^q(X; K_*^M, b)$$

for all $p \leqslant 0$. However as the simplexes collapse Δ^{-p} in the above complex, we discern from the definition of the boundary maps d_1^{pq} that they are trivial if p is odd and isomorphisms if p is even. Therefore we find the second sheet to be

$$E_2^{pq} = \begin{cases} A^q(X; K_*^M, b) & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the spectral sequence converges and we may write

$$E_1^{pq} \Rightarrow A^*(X; K_*^M, b). \tag{8.2}$$

We get the same result if we truncate the diagram (8.1) at $X \times \Delta^N$ for N even. The right-hand side of (8.2) is the target of our desired cycle map. We will use an auxiliary spectral sequence $\widetilde{E}_r^{pq}$ which maps into E_r^{pq}.

Let $\widetilde{A}^a(X \times \Delta^p; K_*^M, b) = \varinjlim A^a_{|Z|}(X \times \Delta^p; K_*^M, b)$, where the limit is over $z^b(X, p)$ as used in the definition of the Chow groups and $|Z|$ denotes the support of Z. If we truncate again at some large even N to avoid convergence problems, we get in the same manner as above another spectral sequence with the first sheet

$$\widetilde{E}_1^{pq} = \begin{cases} \widetilde{A}^q(X \times \Delta^{-p}; K_*^M, b) & \text{for } -p \leqslant N \\ 0 & \text{otherwise.} \end{cases}$$

The natural morphism of cohomology groups from cohomology with supports to the regular one induces a map of spectral sequences

$$\widetilde{E}_1^{pq} \to E_1^{pq}.$$

Using Bloch's notation let $t_N z^b(X,\cdot)$ be the truncation of the complex $z^b(X,\cdot)$ in degree N. Then the cycle class as described in the list above yields a morphism of complexes

$$t_N z^b(X,\cdot) \to \widetilde{E}_1^{\cdot,b}. \tag{8.3}$$

Note that as the limit in the definition of $\widetilde{A}^a(X \times \Delta^p; K_*^M, b)$ runs over cycles of pure codimension b the weak purity axiom implies that $\widetilde{E}_1^{pa} = \widetilde{A}^a(X \times \Delta^p; K_*^M, b) = 0$ for $a < b$. Consequently this holds even for all sheets, i.e., $\widetilde{E}_r^{pa} = 0$ for $a < b$. In particular, for $r > 1$ this implies that the boundary maps

$$d_r^{pb} : \widetilde{E}_r^{p,b} \to \widetilde{E}_r^{p+r,b-r+1} = 0$$

are zero as well. Taking cohomology on both sides of (8.3) and using the fact that the Chow groups derived from the untruncated complex $z^b(X,\cdot)$ maps into the truncated ones we get for any n

$$\mathrm{CH}^b(X,n) \to \widetilde{E}_2^{-n,b} \to \widetilde{E}_\infty^{-n,b}. \tag{8.4}$$

Again by the weak purity axiom we see that $\widetilde{E}_\infty^{p,a} = 0$ for $a < b$. Thus the morphism (8.4) maps in fact into the limit of the $\widetilde{E}_1$ spectral sequence in degree $b - n$. The morphism of spectral sequences $\widetilde{E}_1^{pq} \to E_1^{pq}$ induces then that (8.4) also maps into the limit of the E_1 spectral sequence in degree $b-n$, which is $A^{b-n}(X; K_*^M, b)$ as shown above. This concludes the construction and we get

Corollary 8.2. *For a smooth scheme X/k there is a family of cycle classes*

$$\eta_M^{bn} : \mathrm{CH}^b(X,n) \to A^{b-n}(X; K_*^M, b) = \mathrm{H}^{b-n}(X, \mathscr{K}_b^M). \tag{8.5}$$

We list some properties of the cycle class map for the Milnor K-sheaf.

Normalisation. The class of X in the Chow ring $\mathrm{CH}^*(X,*)$ maps to the identity in the ring $\mathrm{H}^*(X, \mathscr{K}_*^M)$. Indeed, we see that the cycle $[X] \in \mathrm{CH}^0(X,0)$ is mapped via the cycle map to the class of X in $\mathrm{H}^0_X(X, \mathscr{K}^M_0) = \mathrm{H}^0(X, \mathscr{K}^M_0)$, which is isomorphic to $\mathbb{Z}$ and $[X]$ corresponds to the identity as we have seen above.

Functoriality with respect to flat pull-back and proper push-forward. Both the Chow ring and the cohomology of the Milnor K-sheaf are contravariant functorial with respect to flat pull-backs. Let $f : X' \to X$ be flat (a condition that can be dropped in case X is smooth). Then Bloch shows in [Blo86a, Prop.(1.3)] that the complex that calculates the Chow groups is contravariant with respect to f, and consequently there is a well-defined pull-back map

$$f^* : \mathrm{CH}^b(X, n) \to \mathrm{CH}^b(X', n).$$

Likewise Rost constructs in [Ros96, Section 12] a pull-back map

$$f^* : A^p(X; K^M_*, q) \to A^p(X'; K^M_*, q)$$

coming from the corresponding pull-back map on the complex $C^p(X; K^M_*, q)$. The cycle class $[Y] \in \mathrm{H}^b(X, \mathscr{K}^M_b)$ for subschemes $Y \subset X$ of pure codimension which plays an important role in the construction of the cycle class maps are contravariant functorial for morphisms $f : X' \to X$ which preserve the codimension. Thus, if we assume that f is faithfully flat, we obtain functoriality of the cycle class maps η^{bn}_M in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathrm{CH}^b(X, n) & \xrightarrow{\eta^{bn}_M} & \mathrm{H}^{b-n}(X, \mathscr{K}^M_b)) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{CH}^b(X', n) & \xrightarrow{\eta^{bn}_M} & \mathrm{H}^{b-n}(X', \mathscr{K}^M_b) \end{array}$$

Even though we dispose in both cases of push-forwards for a proper morphism $f : X' \to X$, it is not clear to us yet how to make use of it for the cycle class map, as Bloch points out that in case of the Chow groups the push-forward f_* causes a shift in codimension by the degree of f ([Blo86a, Prop. (1.3)]) which according to Rost [Ros96, 3.5] does not occur for his cycle complexes.

Ring homomorphism. It is clear that the cycle class map is additive by linearity. In fact we have the following diagram

$$\begin{tikzcd}
\mathrm{CH}^b(X,n) \otimes \mathrm{CH}^b(X,n) \arrow[r, "\eta_M^{bn} \otimes \eta_M^{bn}"] \arrow[d] & \mathrm{H}^{b-n}(X, \mathscr{K}_b^M) \otimes \mathrm{H}^{b-n}(X, \mathscr{K}_b^M) \arrow[d] \\
\mathrm{CH}^b(X \times X, n) \arrow[r, "\eta_M^{b,n}"] \arrow[d, "\Delta^*"'] & \mathrm{H}^{b-n}(X \times X, \mathscr{K}_b^M) \arrow[d, "\Delta^*"] \\
\mathrm{CH}^b(X,n) \arrow[r, "\eta_M^{b,n}"] & \mathrm{H}^{b-n}(X, \mathscr{K}_b^M)
\end{tikzcd}$$

where the upper square commutes by linearity and the lower one by pulling back along the diagonal. This extends of course linearly to addition of cycles of different codimension and degree.

Multiplicativity requires more work. Multiplication in the higher Chow ring is described by Bloch in [Blo86a, Section 5]. In order to do this, it is sufficient to construct a map in the derived category for the corresponding complexes. More precisely, let X and Y be quasi-projective algebraic k-schemes. Then Bloch constructs a map

$$s\left(z^a(X,\cdot) \otimes z^b(Y,\cdot)\right) \to z^{a+b}(X \times Y, \cdot),$$

where on the left-hand side s denotes the simple complex associated with a double complex. The idea is to fix a triangulation for $\Delta^m \times \Delta^N \cong \mathbb{A}^{m+n}$ for all m, n such that it induces a well-defined morphism on the complexes. A triangulation is a family $T = \{T_{m,n}\}_{m,n\in\mathbb{N}}$ with

$$T_{m,n} = \operatorname{sgn}(\theta)\theta,$$

where θ is a face map $\Delta^{m+n} \to \Delta^m \times \Delta^n$. It is possible to fix a system of maps T such that it induces a morphism of complexes

$$s\left(z^*(X,\cdot) \otimes z^*(Y,\cdot)\right) \to z^*(X \times Y, \cdot),$$

where it is defined. However, the problem hereby is that $T_{n,m}\left(z^*(X,\cdot) \otimes z^*(Y,\cdot)\right)$ is not necessarily contained in $z^*(X \times Y, \cdot)$ as images of cycles might not meet all faces properly. The solution is to take the subcomplex of $s\left(z^*(X,\cdot) \otimes z^*(Y,\cdot)\right)$

generated by products $Z \otimes W$ such that Z and W are irreducible subvarieties of $X \times \Delta^m$ and $Y \times \Delta^n$, respectively, and such that $Z \times W \subset X \times Y \times \Delta^m \times \Delta^n$ meets all faces properly. We denote this subcomplex by $z^*(X, Y, \cdot)' \subset s\,(z^*(X, \cdot) \otimes z^*(Y, \cdot))$. Bloch shows in [Blo86a, Theorem 5.1] that this inclusion is in fact a quasi-isomorphism. As a consequence, one obtains a commutative diagram

$$\begin{tikzcd} s(z^*(X,\cdot) \otimes z^*(Y,\cdot)) \arrow[r, "\sim"] \arrow[drr] & z^*(X,Y,\cdot)' \arrow[r, "T"] & z^*(X \times Y, \cdot) \arrow[d] \\ & & z^*(X,\cdot) \end{tikzcd}$$

in the derived category and this induces an action of $\mathrm{CH}^*(Y, \cdot)$ on $\mathrm{CH}(X, \cdot)$. In particular, if $Y = X$ is smooth, one obtains a product on $\mathrm{CH}^*(X, \cdot)$ via pull-back along the diagonal

$$\mathrm{CH}^a(X, n) \otimes \mathrm{CH}^b(X, m) \to \mathrm{CH}^{a+b}(X \times X, n + m) \xrightarrow{\Delta^*} \mathrm{CH}^{a+b}(X, n + m),$$

which makes it into an anticommutative ring [Blo86b, Corollary 5.7].

By the above statements, it is sufficient, in order to see if the family of maps η_M^{bn} is compatible with products, to consider the subcomplex $z^*(X, X, \cdot)' \subset s\,(z^*(X, \cdot) \otimes z^*(X, \cdot))$. Thus let $Z \in z^a(X, n)$ and $W \in z^b(X, m)$ be irreducible subvarieties of $X \times \Delta^n$ and $X \times \Delta^m$, respectively, such that $Z \times W \subset X \times X \times \Delta^n \times \Delta^m$ meets all faces of $\Delta^n \times \Delta^m$ properly, which means that $Z \otimes W$ is in the set of generators of $z^*(X, X, \cdot)'$. The cycle class of Milnor K-theory mentioned above sends the class of Z to a unique class $[Z] \in A^a_Z(X \times \Delta^n, a) = A^0(Z; K^M_*, 0) \cong \mathbb{Z}$ and W to a unique class $[W] \in A^b_W(X \times \Delta^m, b) = A^0(W; K^M_*, 0) \cong \mathbb{Z}$, which in both cases represents the identity. Rost's definition of (cross) products for cycle modules in [Ros96, Section 14]

$$C^p(Y; N, n) \times C^q(X, M, m) \to C^{p+q}(Y \times X; M)$$

holds in particular for the case of $N = M = K^M_*$. In this case the product is anticommutative as shown in [Ros96, Corollary 14.3]. Hence the product of $[Z]$ and $[W]$ as evoked above can easily be given as

$$\begin{aligned} A^0(Z;K_*^M,0)\times A^0(W;K_*^M,0) &\rightarrow A^0(Z\times W;K_*^M,0)\\ [Z]\times[W] &\mapsto [Z\times W] \end{aligned}$$

as all cycles involved represent the identity. Thus by means of the corresponding inclusions we obtain a commutative diagram

$$\begin{tikzcd} z^a(X,n)\otimes z^b(X,m) \arrow[r] \arrow[d] & \widetilde{A}^a(X;K_*^M,n)\otimes\widetilde{A}^b(X;K_*^M,m) \arrow[d] \\ z^{a+b}(X\times X,n+m) \arrow[r] & \widetilde{A}^{a+b}(X\times;K_*^M,n+m) \end{tikzcd}$$

This shows that the morphism of complexes (8.3) is compatible with products and since this is the core of Bloch's construction of cycle class maps, they are compatible with products as well and one has a diagram

$$\begin{tikzcd} \mathrm{CH}^a(X,n)\otimes\mathrm{CH}^b(X,m) \arrow[r, "\eta_M^{an}\otimes\eta_M^{bm}"] \arrow[d] & \mathrm{H}^{a-n}(X,\mathscr{K}_a^M)\otimes\mathrm{H}^{b-m}(X,\mathscr{K}_b^M) \arrow[d] \\ \mathrm{CH}^{a+b}(X\times X,n+m) \arrow[r, "\eta_M^{a+b,n+m}"] \arrow[d, "\Delta^*"'] & \mathrm{H}^{a+b-n-m}(X\times X,\mathscr{K}_{a+b}^M) \arrow[d, "\Delta^*"] \\ \mathrm{CH}^{a+b}(X,n+m) \arrow[r, "\eta_M^{a+b,n+m}"] & \mathrm{H}^{a+b-n-m}(X,\mathscr{K}_{a+b}^M) \end{tikzcd}$$

where the upper square commutes due to the discussed reasons and the lower one again by pulling back along the diagonal.

8.3 Higher cycle classes with integral coefficients in the overconvergent complex

We now use the map (5.2) of section 5.7

$$d\log^n : \mathscr{K}_n^M \rightarrow W^\dagger\Omega[n]$$

to define higher cycle classes with coefficients in the overconvergent cohomology theory. Remember that it induces a morphism of cohomology groups

$$\mathrm{H}^m(X,\mathscr{K}_i^M)\rightarrow\mathbb{H}^{m+i}(X,W^\dagger\Omega).$$

Note that whereas the first cohomology theory is bigraded, the second one is not. However, by definition the image of $d\log$ lies in the truncated complex $W^\dagger\Omega^{\geqslant n}[n]$, which is a subcomplex of $W^\dagger\Omega[n]$. As a consequence, $d\log$ factors and we can write

$$d\log^n : \mathscr{K}_n^M \to W^\dagger\Omega[n],$$

and therefore on the cohomological level a morphism

$$\mathrm{H}^m(X, \mathscr{K}_i^M) \to \mathbb{H}^{m+i}(X, W^\dagger\Omega^{\geqslant i}).$$

Then the cycle class map for the Milnor K-sheaf (8.5) induces the following result

Proposition 8.3. *For $b, n \geqslant 0$ there exist cycle class maps*

$$\eta_{\mathrm{sc}}^{bn} : \mathrm{CH}^b(X, n) \to \mathbb{H}^{2b-n}(X, W^\dagger\Omega^{\geqslant b}).$$

By functoriality of the morphism of cohomology rings

$$\mathrm{H}^*(X, \mathscr{K}_*^M) \to \mathbb{H}^*(X, W^\dagger\Omega^{\geqslant *})$$

the cycle classes η_{sc}^{bn} satisfy similar properties as mentioned above for the cycle classes η_M^{bn}.

APPENDIX A

COMPARISON OF THE FIRST RIGID AND OVERCONVERGENT CHERN CLASSES FOR PROJECTIVE SPACE

We consider the projective space $\mathbb{P} = \mathbb{P}^n_k$ over k and compare the image of the class of $c_1^{\mathrm{sc}}(\mathscr{O}_{\mathbb{P}}(1))$ under the canonical injection $\mathbb{H}^i(\mathbb{P}, W^\dagger\Omega) \to \mathbb{H}^i(\mathbb{P}, W^\dagger\Omega) \otimes \mathbb{Q} \cong \mathrm{H}^i_{\mathrm{rig}}(\mathbb{P}\,/K)$ with the class of $c_1^{\mathrm{rig}}(\mathscr{O}_{\mathbb{P}}(1))$.

Lemma A.1. *The image of $c_1^{rig}(\mathscr{O}_{\mathbb{P}}(1))$ under the comparison isomorphism*

$$\mathrm{H}^2_{rig}(\mathbb{P}\,/K) \to \mathbb{H}^2(\mathbb{P}, W^\dagger\Omega_{\mathbb{P}}) \otimes \mathbb{Q}$$

coincides with $c_1^{\mathrm{sc}}(\mathscr{O}_{\mathbb{P}}(1))$.

Proof. The class of $c_1^{\mathrm{sc}}(\mathscr{O}_{\mathbb{P}}(1))$ can be described in terms of Čech cocycles in a natural way. If we think of $\mathbb{P}$ as $\mathrm{Proj}(k[x_0, \ldots, x_n]$, let $H_i = \{x_i = 0\}$ and $U_i = \mathbb{P} - H_i$. This open set is isomorphic to the affine $\mathrm{Spec}\,(A_i)$, where $A_i = k[u_{0i}, \ldots, \hat{u}_{ii}, \ldots, u_{ni}]$ with $u_{ji} = \frac{x_j}{x_i}$ (the hats in this context always means that the variable is omitted). The open sets $\mathfrak{U} = \{U_i\}$ form a Čech cover of $\mathbb{P}$ that trivialises the twisting sheaf $\mathscr{O}_{\mathbb{P}}(1)$. According to [Gil81, Definition 2.3], the first Chern class of $\mathscr{O}_{\mathbb{P}}(1)$ in $\mathrm{H}^1(\mathbb{P}, \mathscr{K}_1^M)$ is calculated via the Čech cochain given as

$$u_{ij} \qquad \text{on } U_i \cap U_j.$$

It is easily checked that this is a cocycle. The image of this under $d\log : \mathscr{K}_1^M \to W^\dagger\Omega_{\mathbb{P}}\,[1]$ is

$$d\log u_{ij} = \frac{d[u_{ij}]}{[u_{ij}]},$$

which defines a Čech cocycle in $Z^2(\mathfrak{U}, W^\dagger\Omega_{\mathbb{P}})$.

This suffices on the overconvergent side. Now we have to give the class of the twisting sheaf on the rigid side. Let $\mathbb{P}^n_{W(k)}$ be the associated formal scheme over $\mathrm{Spf}(W(k))$. The cover of $\mathbb{P}^n_{W(k)}$ that reduces to $\mathfrak{U}$ on $\mathbb{P}$ is given by

$$\mathscr{U}_i = \mathrm{Spf}\left(\mathscr{A}_i\right),$$

where $\mathscr{A}_i = W(k)\{u_{0i}, \dots, \hat{u}_{ii}, \dots, u_{ni}\}$. Denote by $\mathfrak{U}_K$ the induced cover on the rigid generic fibre, which consists of the affinoid subsets associated to the Tate algebras $K\langle u_{0,i}, \dots, \hat{u}_{ii}, \dots, u_{ni}\rangle$. Following the construction of Petrequin in [Pet03, Section 3.1], the first Chern class of $\mathscr{O}_{\mathbb{P}}(1)$ in $\mathrm{H}^2_{\mathrm{rig}}(\mathbb{P}\,/K)$ is calculated by the Čech cocycle in $Z^2(\mathfrak{U}_K, \Omega_{]\mathbb{P}[})$ given by

$$\frac{du_{ij}}{u_{ij}} \qquad \text{on } \mathscr{U}_i \cap \mathscr{U}_j.$$

The remaining step is to relate the two classes under the comparison morphism.

A Witt frame for U_i can be given by

$$(U_i, F_i, \varkappa) = (\mathrm{Spec}(A_i), \mathrm{Spec}(B_i), \varkappa),$$

where $B_i = W(k)[u_{0i}, \dots, \hat{u}_{ii}, \dots, u_{ni}]$ and $\varkappa : B_i \to W\left(A_i\right)$ sends u_{ji} to its Teichmüller lift. Indeed, let $B_i^\dagger$ be the associated dagger algebra. In this case, Davis has shown in [Dav09, Proposition 2.2.2] that if we choose the Frobenius lift to be $F(u_{ij}) = u_{ij}^p$ on the coordinates, the induced comparison map sends u_{ij} to its Teichmüller lift. This descends to B_i, and it is clear that the diagram

$$\begin{tikzcd} & W\left(A_i\right) \arrow[d, "\mathrm{proj}"] \\ B_i \arrow[ur, "\varkappa"] \arrow[r] & A_i \end{tikzcd}$$

commutes, so the given datum indeed is a Witt frame. By reason that the variables map to their Teichmüller lifts, which we have seen to be overconvergent, it is evident that the image of $\varkappa$ is contained in $W^\dagger(A_i)$.

Since we assumed that k is perfect, the kernel I of $B_i \to A_i$ is monogenic and $I = (p)$. Let R_i be as above the completion of B_i with respect to I. As we mentioned in the previous section, to give a morphism

$$\Gamma\left(]U_i[_{\hat{F}_i}, \mathscr{O}_{]U_i[_{\hat{F}_i}}\right) \to W(A_i) \otimes \mathbb{Q}$$

it is enough to give a compatible system of morphisms

$$R_{i,n} \to W(A_i),$$

where

$$R_{i,n} = R_i[T]/(p^n - pT).$$

Under $\varkappa$ the image u_{ju} is its Teichmüller lift. Furthermore, by the construction of Davis, Langer and Zink mentioned above, T will be sent to $\frac{1}{p}\varkappa(p^n)$. This gives a compatible system of maps and determines the desired morphism. In particular, we see that the variables u_{ji} that appear in the formal algebra as well as in the associated affinoid algebra are invariably sent to $[u_{ji}]$. By the universal property of Kähler differentials, this extends to $\Gamma\left(]U_i[_{\hat{F}_i}, \Omega_{]U_i[_{\hat{F}_i}}\right) \to W\Omega_{U_i} \otimes \mathbb{Q}$ and in particular it means that

$$\frac{du_{ji}}{u_{ji}} \mapsto \frac{d[u_{ji}]}{[u_{ji}]},$$

up to multiplication by $\frac{1}{p}$.

To extend this morphism to a morphism

$$R\Gamma_{\text{rig}}(X/K) \to W^{\dagger}\Omega_{X/k} \otimes \mathbb{Q},$$

we need to take into account strict neighbourhoods of $]U_i[_{\hat{F}_i}$ in $F_{i,K}^{\text{an}}$. A system of strict neighbourhoods is given by open subsets $U_{\lambda,\eta}$ in the sense of rigid geometry with $\lambda = p^{-\frac{1}{v}}$ and $\eta = p^{-\frac{1}{r}}$ given in terms of affinoid algebras as

$$\mathcal{T}_i = K\langle \lambda u_{oi}, \ldots, \hat{u}_{ii}, \ldots, \lambda u_{ni}, T\rangle/(p^r - pT).$$

It consists of all power series

$$\sum a_{I,j}\underline{u}^I T^j\ , \qquad a_{I,j} \in K,$$

such that $\lim_{|I|+j\to\infty} |a_{I,j}| \left(\frac{1}{\lambda}\right)^{|I|} = 0$. Thus clearly the single elements u_{ji} are contained in this algebra. Moreover, since the morphism

$$\mathcal{T}_i \to W^{\dagger}(A_i) \otimes \mathbb{Q}$$

is constructed in a compatible way with the one described in the previous paragraph, we see that u_{ij} is still mapped to the Teichmüller lift $[u_{ij}]$.

The procedure given here induces for each $0 \leqslant i \leqslant n$ a natural morphism

$$R\Gamma_{\mathrm{rig}}(U_i/K) \to W^{\dagger}\Omega_{U_i/k} \otimes \mathbb{Q}\,.$$

In particular it sends the Čech cocycle given over $U_i \cap U_j$ by $\frac{du_{ij}}{u_{ij}}$ to its Teichmüller lift.

In order to glue the pieces together one passes to dagger spaces. The space $]U_i[_{\hat{F}_i}$ is covered by affinoid opens H_{η_u}, $\eta_u = p^{-\frac{1}{u}}$, with corresponding algebras

$$C_{\eta_u} = K\langle u_{0i}, \dots, \hat{u}_{ii}, \dots, u_{ni}, S\rangle/(p^u - pS).$$

To endow $]U_i[_{\hat{F}_i}$ with a dagger space structure as discussed in [GK00] we replace the C_{η_u} over a suitable extension $\tilde{K}/K$ by

$$\tilde{K}\langle u_{0i}, \dots, \hat{u}_{ii}, \dots, u_{ni}, S\rangle/(p - p^{1-u}S),$$

which is clearly isomorphic over $\tilde{K}$. Moreover, for $t > u$ and arbitrary $\lambda = p^{-\frac{1}{v}}$ there is an open immersion

$$H_{\eta_u} \to U_{\lambda,\eta_t}$$

given by an algebra homomorphism over $\tilde{K}$

$$\tilde{K}\langle \lambda u_{0i}, \dots, \hat{u}_{ii}, \dots, \lambda u_{ni}, T\rangle/(p - p^{\frac{1}{t}}T) \to \tilde{K}\langle u_{0i}, \dots, \hat{u}_{ii}, \dots, u_{ni}, S\rangle/(p - p^{1-u}S),$$

sending λu_{ji} to $p^{\frac{1}{v}} u_{ji}$ and T to $p^{\frac{1}{u}-\frac{1}{t}}S$. It maps H_{η_u} to the interior U_{λ,η_t}, and in this way one obtains a dagger space structure by letting λ and η vary.

It is possible to rewrite the morphisms $\Gamma_{\mathrm{rig}}(U_i/K) \to W^{\dagger}\Omega_{U_i/k} \otimes \mathbb{Q}$ in terms of dagger spaces:

$$\Gamma\left(]U_i[^{\dagger}_{\hat{F}_i}, \Omega_{]U_i[^{\dagger}_{\hat{F}_i}}\right) \to W^{\dagger}\Omega_{U_i/k} \otimes \mathbb{Q},$$

where it is to notice that u_{ji} is still mapped to $[u_{ji}]$.

Using the simplicial scheme associated to the covering $\mathbb{P} = \bigcup U_i$ the above construction induces a natural quasi-isomorphism

$$R\Gamma_{\mathrm{rig}}(X/K) \to R\Gamma(X, W^{\dagger}\Omega X/k) \otimes \mathbb{Q},$$

which sends the Čech cocycle of rigid cohomology

$$\left(\frac{du}{u}\right) \in Z^2(\mathfrak{U}_K, \Omega_{]\mathbb{P}[}))$$

to the Čech cocycle of overconvergent cohomology

$$\left(\frac{d[u]}{[u]}\right) \in Z^2(\mathfrak{U}, W^{\dagger}\Omega_{\mathbb{P}}).$$

In particular this makes it evident that the first rigid and overconvergent Chern classes of $\mathscr{O}_{\mathbb{P}}(1)$ are compatible. $\square$

APPENDIX B

IS $\mathrm{H}^N(X, \mathscr{K}_I^M)$ A TWISTED DUALITY THEORY?

Let X/k be smooth and $\mathscr{K}_*^M$ the Milnor K-sheaf (the usual or improved version according to the context). In Theorem 4.10 we verified some of the properties suggested by Gillet for the duality theory defined by $\Gamma(i) = \mathscr{K}_i^M$. We want to check which of Gillet's axioms apply to this theory additionally.

1. **Homology functor.** It is clear that by definition of the Chow groups for cycle modules for dimension we dispose of a covariant functor

$$X \to \bigoplus_{\substack{i\geqslant 0\\ j\in\mathbb{Z}}} A_i(X; K_*^M, j)$$

into the category of bigraded abelian groups. Let f, g be proper, and i, i' open immersions such that the square

$$\begin{tikzcd} U \arrow[r, "i'"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "i"'] & Y \end{tikzcd}$$

commutes. As f and g are proper, the definition of push-forward makes sense. Note also, that the open immersions have dimension $s = 0$, and the pull-back maps are defined as $i^* = [\mathscr{O}_V, i, 0]$ and $i'^* = [\mathscr{O}_U, i', 0]$. Then point (3) from Proposition 3.8 shows that the diagram

$$\begin{tikzcd} A_i(U; K_*^M, j) \arrow[d, "g_*"'] & A_i(X; K_*^M, j) \arrow[l, "i'^*"'] \arrow[d, "f_*"] \\ A_i(V; K_*^M) & A_i(Y; K_*^M, j) \arrow[l, "i^*"] \end{tikzcd}$$

commutes.

2. **Localisation sequence.** This is the long exact sequence of homology (3.2) for Chow groups, where the boundary map is induced by the boundary map on cycle complexes (Point 4. of the list of morphisms).

3. **Cap product.** Recall that there is a pairing of cycle modules

$$K_*^M \times K_*^M \to K_*^M,$$

which respects grading. Using the map "multiplication with units" from point 3 in Subsection 3.2 this induces a pairing of complexes

$$C_p(X, K_*^M, j) \times C_Y^q(X, K_*^M, i) \to C_{p-q}(Y, K_*^M, j-i),$$

where we have used that $C^q = C_{n-q}$ as X is of dimension n and where C_Y^q means sections with support in Y. This map respects the grading on K_*^M since the original pairing on K_*^M does so. Moreover, it respects the grading in dimension as it is a generalised correspondence map mentioned in [Ros96, (3.9)]. Applying the (co)homology functor, we obtain a pairing

$$\bigcap : A_p(X; \mathrm{K}_*^M, j) \otimes A_Y^q(X; K_*^M, i) \to A_{p-q}(Y; K_*^M, j-i).$$

Consider the Cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f_Y"'] & X \arrow[d, "f_X"] \\ Y' \arrow[r] & X' \end{tikzcd}$$

as in 3. If we assume in addition that f_X is flat, Lemma 3.9 tells us that for an element a on X

$$f_* \circ \{a\} \circ f^* = \{\tilde{f}_*(a)\}.$$

This implies (still under the assumption that f is flat) that for $\alpha \in \mathrm{H}_p(X \mathscr{K}_j^M)$ and $z \in \mathrm{H}_Y^q(X', \mathscr{K}_i^M)$ we have

$$f_*\alpha \cap f^!(z)) = f_*(\alpha) \cap z$$

i.e.,the projection formula, however in a less general setting than proposed by Gillet.

4. **Fundamental class.** Let X be of relative dimension $\leqslant n$ over k. By definition we have that
$$\mathrm{H}_n(X, \mathscr{K}_n^M) \cong \mathrm{H}^0(X, \mathscr{K}_0^M)$$
is a quotient of $\coprod_{x \in X^{(0)}} K_0(x)$. Since $\mathscr{K}_0^M(X) = \mathbb{Z}$ the fundamental class corresponds to the class [1].

5. **Poincaré duality.** Assume that X is of relative dimension n and $Y \to X$ a closed immersion. We see easily that
$$C_Y^{n-i}(X; K_*^M, n+r) = \coprod_{\substack{x \in X^{(n-i)} \\ x \in Y}} K_{r+i}^M(x)$$
and
$$C_i(Y; K_*^M, r) = \coprod_{x \in Y_{(i)}} K_{r+i}^M(x)$$
coincide and as a consequence we get an isomorphism
$$A_Y^{n-i}(X; K_*^M, n+r) \to A_i(Y; K_*^M, r),$$
which represents a deviation in the second index from Gillet's fifth axiom. However, the formula
$$A_n(X; K_*^M, n) \cong A^0(X; K_*^M, 0)$$
still holds.

6. **Sections with support.** Let $j : Y \to X$ be a closed immersion of codimension $c = p$. Analogue to the reasoning in the previous point, we have
$$\begin{aligned} C_Y^{i+p}(X; K_*^M, r+p) &= \coprod_{\substack{x \in X^{(i+p)} \\ x \in Y}} K_{r-i}^M(x) \\ &= \coprod_{x \in Y^{(i)}} K_{r-i}^M(x) = C^i(Y; K_*^M, r), \end{aligned}$$
which gives the desired formula—however, only if the codimension c coincides with the shift in cohomology p.

7. **Projection formula.** Recall that the Milnor K-sheaf is in view of the Gersten conjecture (cf. Corollary 2.10) defined as a quotient of the tensor algebra. Therefore it is clear that the projection formula from (3) can be captured in a commutative diagram of the form

$$\begin{tikzcd}
Rj_! \mathscr{K}^M_r|_Y \otimes^L_{\mathbb{Z}} \mathscr{K}^M_s|_X \arrow[r, "1\otimes j^!"] \arrow[d, "j_!\otimes 1"', "\sim"] & Rj_! \left(\mathscr{K}^M_r|_Y \otimes^L_{\mathbb{Z}} \mathscr{K}^M_s|_Y\right) \arrow[d] \\
Rj^! \mathscr{K}^M_{r+p}|_X[p] \otimes^L_{\mathbb{Z}} \mathscr{K}^M_s|_X \arrow[dr] & Rj_! \left(\mathscr{K}^M_{r+s}|_Y\right) \arrow[d, "\sim"] \\
 & Rj^! \left(\mathscr{K}^M_{s+r+p}|_X[p]\right)
\end{tikzcd}$$

8. **Cross product.** As we mentioned in 3.17, Rost defines a cross product for cycle modules. The fact that it is defined pointwise implies that it can be defined for quasi-projective schemes over k.

9. **Homotopy invariance.** This is a special case of the homotopy invariance (3.4) for Chow groups that follows from Axiom (**H**) in the definition of cycle modules.

10. **Projective bundle formula.** We proved this in Proposition 3.19

11. **Cycle class map.** This is clear from the definition of the first Milnor K-group. Indeed, recall that by definition of the Milnor K-sheaf

$$\mathscr{K}^M_1 = \mathscr{O}^*_X,$$

and the well known isomorphism for a scheme X

$$\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathscr{O}^*_X)$$

gives a natural transformation of contravariant functors on the big Zariski site $\mathscr{V}$.

In conclusion, one can say, that the duality theory defined by $\Gamma(i) = \mathscr{K}^M_i$ is not quite as general as Gillet's twisted duality theory, but for many practical purposes this is sufficient.

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